TWO DIMENSIONAL AND AXI-SYMMETRIC THERMAL AND STRESS ANALYSIS BY TRANSFER FEM

By

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DEPARTMENT OF NUCLEAR ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR

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P. Mun.

CERTIFICATE

This is to certify that the present work "Two dimensional and axi-symmetric thermal and stress analysis by transfer FEM", has been carried out by Mr. SANJAY DWIVEDI under our supervision and it has not been submitted elsewhere for a degree.

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ABSTRACT

In the present work transfer finite element equations are developed for a four noded rectangular element and an eight noded isoparametric element. These formulations are next used for solving two dimensional and axi-symmetric thermal and stress analysis problems employing transfer FEM. A rectangular domain and a pressure vessel with hemi-spherical heads is considered for two dimensional and axi-symmetric problems respectively. Results are obtained for different geometrical and natural boundary conditions in both two dimensional and axi-symmetric problems.

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SANJAY DWIVEDI

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NOTATION

$[\ ]$	Rectangular or square matrix
$\{\ \}$	Column matrix
$\lfloor\ \rfloor$	Row matrix
$[\ ]^T$	Matrix transpose
$[\ ]^{-1}$	Matrix inverse
A	Area or cross-sectional area
$[B]$	Spatial derivatives of the field variables
$[C]^{(e)}$	Elemental thermal capacitance matrix
$\{d\}$	Displacement matrix
$[E]$	Matrix of elastic stiffnesses
$[k]$	Stiffness matrix
$[k_t]^{(e)}$	Elemental thermal stiffness matrix
$[k_h]^{(e)}$	Convective coefficient matrix
$[N]$	Shape (or basis or interpolation) functions
T	Temperature
$\{T\}^{(ne)}, \{\theta\}^{(ne)}$	Elemental temperature matrix
T_f	Fluid temperature
θ	Temperature difference $(T - T_f)$
u, v	Displacements in directions x,y (or r,z)
$\{q\}$	Heat flux vector
$ql\|_{ij}$	Heat flux through edge ij

$Fx\|_{ij}, Fy\|_{ij}, Fr\|_{ij}, Fz\|_{ij}$	Forces along x, y, r, and z directions respectively
$\{p\}$	Load matrix
$\{F\}$	Force matrix
h	Heat transfer coefficient
$\{\epsilon\}$	Strains
$\{\sigma\}$	Stresses
t	Time
ρ	Mass density
c	Specific heat of material
$[TFM]_m$	Transfer matrix of strip m
$[t]^{(e)}$	$\left([k_t]^{(e)} + [k_h]^{(e)}\right)$ matrix
[z]	Final transfer matrix

CHAPTER - 1

INTRODUCTION

Exact methods fail to provide a solution for most of the real life engineering problems. This is more true when components involved are non-uniform. Approximate methods have to be used for obtaining reasonably accurate solutions for such problems. Such methods suffer from the limitations of satisfying the geometrical and natural boundary conditions right in the beginning. To overcome these limitations numerical methods like finite difference, transfer matrix method and finite element methods have been developed. In the present work the latter two are relevant and are discussed here.

Transfer matrix method was the first one to be developed. Its earliest application was to the torsional vibrations of shafts by Holzer [1] in 1921. In 1944, after twenty three years it was extended to beam vibrations by Myklestad [2]. It was further developed by many research workers including Pestel and Leckie [3] who wrote the text [3] in 1963. Large references are given in this text [3]; and problems like vibrations of Timoshenko beams, circular arches, turbine generator shafts, shear panels etc. are studied. Method is also applied to static problems of beams, rotating disks and stability problems of beams. It may be noted that ninety nine percent of this text is

devoted to one dimensional problems. They also tried to extend this transfer matrix approach to plate vibrations by replacing the plate by an equivalent network of beams based on the Hernnikoff model. This type of modelling is very artificial and questionable. A more general procedure for getting transfer matrices for plate and shell problems is given by Pestel [4].

Next text, Pilkey and Chang [5], dealing with this transfer matrix came after fifteen years i.e. in 1978. Probably the last and latest to come is Tesar and Fillo [6] in 1988. These texts deal with plate problems more extensively. Thus it may be noted that probably only four texts have come in this twentieth century so far.

These transfer matrices are usually obtained from statics or by using results that are tabulated in engineering hand books. They are also obtained from the closed form solutions of the governing differential equations of the problem, but quite a difficult and many a times impossible job. In addition many analytical methods like Laplace transformation, exponential expansion, matrizant, product integral [3,5] and numerical methods [5,6] are used to obtain the transfer matrices.

Since fifties when modern electronic computers became available, finite element method got developed extensively. Now-a-days it is the most widely accepted technique for the real life enginering problems. Probably more than three hundred texts have been written since 1967.

Beauty of these methods lie in the fact that the boundary

conditions are introduced in the end. In transfer matrix methods one multiplies transfer matrices one after the another, so this method does not tax the memory of the computers. But as said earlier, accurate transfer matrices are not that easy to obtain. In finite element method, finite element matrices are added and the size of assembled matrices goes on increasing with the increase in number of elements. Thus one faces serious limitations of computer memory when using this method for more involved problems.

Thus in view of the above mentioned limitations of transfer matrix method and finite element methods, a new approach was developed by Dokainish [7] in 1972 for vibrations of rectangular plates using triangular elements with three nodes (incompatible elements). This approach combines the advantages of both these methods i.e. derivation of finite element matrices is very straight forward (these are changed to transfer matrices in a very simple way) and very small computer memory is needed. This new approach is called transfer finite element method (TFEM) in this work. It may be pointed out that correct FEM equations were not used in this work but correct TFEM equations were obtained using the knowledge of statics. McDaniel and Eversole [8] proposed a similar approach in 1977 for a stiffened plate structure. This method was probably improved by Pilkey [9] in Technical Report in 1979. Sankar and Hoa [10] in 1980 offered an approach in which an extended transfer matrix relating the state vectors which consists of state variables displacements and

forces and their derivatives with respect to frequency were used.

This approach was also independently developed by Gupta [11] in late 82. Correct FEM equations were used to develop transfer matrices. This method was developed for free vibrations of one-dimensional problems of axial rods, Euler beams, Timoshenko beams (Single fourth order differential equation and two simultaneous second order coupled differential equations), rotating Euler and Timoshenko beams and two dimensional problems of rectangular isotropic and anisotropic membranes using rectangular four noded elements. This was also developed for static problems of one dimensional axial rods and Euler beams and two dimensional steady state heat conduction problem in a rectangular region using triangular elements with three nodes. Results obtained by classical approach and TFEM were found to be identical for all practical purposes.

Tesar [12] applied this method to non-linear vibrations of shell structures in 1983,technical report in Slovak,and extended this work [13] in 1985.Ogha and Shigematsu [14] used it in 1983 for bending and buckling problems and developed a technique for treating the structure with intermediate conditions.

TFEM was used by Subramanya [15] for flexibility analysis of pipes in 1985. Pipes having branches and intermediate constraints were taken care of. Various loadings were considered. This method was used by Goel [16] for finding critical speeds of rotors in 1986 and also by Mathew [17] for studying flexural vibrations of rotating shafts in 1992. Mathew

[17] studied flexural vibrations of damped shafts including the effect of shear deformation, rotary inertia and gyroscopic actions using four second order differential equations. Critical speeds, complex natural frequencies and unbalance responses were obtained. All types of geometrical and natural boundary conditions, including bearing cross coupled properties were incorporated exactly.

Gupta [18] used this method in 1987 for vibration analysis of cable networks and beam grillages. Orthogonal and non-orthogonal cable networks were considered. Transverse vibrations of beam grillages in bending and combined-torsional modes with various types of boundary conditions were analysed.

This method has been developed considerably by Tesar [6] and applied to many linear and nonlinear complex problems like thin walled beams, corrugated and sandwich panels, shells, elastic-plastic behaviour, aeroelastic response and so on. Tesar [6] calls this method as finite element transfer matrix (FETM). It may be noted that here also correct FEM equations are not used in this text [6] but correct TFEM equations are obtaied using the knowledge of statics later. In two dimensional problems only four noded elements are considered.

In TFEM, the domain is divided into number of strips. Strip directions are so chosen that minimum number of elements are needed in these strips. Each strip is divided into number of finite elements. Governing finite element matrices are assembled for this strip in the usual way. The boudary conditions of top

and bottom edges are incorporated. These equations are rearranged to obtain the transfer matrix for this strip. These transfer matrices are multiplied, boundary conditions of the left and the right ends are applied, and thus obtaining the first state vector. Other state vectors are obtained in the usual way.

In the present work TFEM equations have been developed for two-dimensional heat conduction and stress analysis problems using four noded rectangular and eight noded isoparametric elements. Convective heat transfer boundary conditions have also been considered. All this is given in Chapter 2.

Above is also done for axi-symmetric heat conduction and stress analysis problems and is given in Chapter 3. Here a case of unequal adjacent strips has also been considered. Method is also extended to transient heat conduction.

To validate the above, results are obtained for different problems and are given in Chapter 4.

Chapter 5 concludes the present work.

CHAPTER - 2
STEADY STATE THERMAL AND STRESS ANALYSIS OF TWO DIMENSIONAL PROBLEMS

In this chapter TFEM equations have been developed from FEM equations for two dimensional steady state heat conduction (with various boundary conditions) and stress analysis problems using four noded rectangular and eight noded isoparametric elements.Convective boundary conditions have also been considered. Transfer matrices have been developed for a typical strip(first strip)of the plane lamina.In the last transfer finite element method (TFEM) has beem illustrated for both thermal and stress analysis problems.

2.1 Governing differential equation, heat conduction

The governing differential equation for heat conduction in a plane with no heat source or sink is [20], Fig. 2.1.1

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T}{\partial y}\right) = 0 \qquad (2.1.1)$$

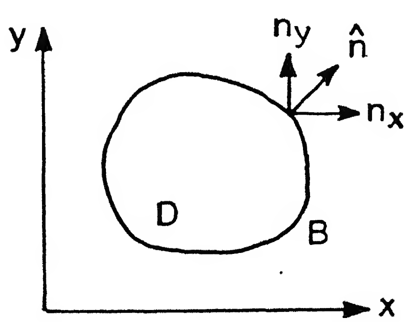


Fig. 2.1.1 Plane lamina with domain D and boundary B

The boundary conditions can be of the type:

$$T = T_p(x,y)$$

or

$$k_x \frac{\partial T}{\partial x} n_x + k_y \frac{\partial T}{\partial y} n_y + q + h(T - T_f) = 0 \qquad (2.1.2)$$

2.2 FEM equations, heat conduction

2.2.1 Without convection

Here the field variable, temperature, is approximated over the element as,

$$T^{(e)} = a + bx + cy + dxy + \ldots\ldots\ldots$$

$$= \lfloor N_1 \; N_2 \ldots\ldots N_r \rfloor \begin{Bmatrix} T_1 \\ T_2 \\ \vdots \\ T_r \end{Bmatrix} = \lfloor N \rfloor \{T\}^{(ne)} \qquad (2.2.1)$$

Substituting this in equation (2.1.1), residue is obtained as,

$$R^{(e)} = \frac{\partial}{\partial x}\left(k_x \frac{\partial T^{(e)}}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T^{(e)}}{\partial y}\right) \qquad (2.2.2)$$

Minimizing this residue by the Galerkin method, one gets

$$\iint_{A^{(e)}} N_i \left[\frac{\partial}{\partial x}\left(k_x \frac{\partial T^{(e)}}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial T^{(e)}}{\partial y}\right)\right] dx\,dy = 0 \qquad (2.2.3)$$

Doing intergration by parts, one gets

$$\iint_{A^{(e)}} \left(N_{i,x}\, k_x \frac{\partial T^{(e)}}{\partial x} + N_{i,y}\, k_y \frac{\partial T^{(e)}}{\partial y}\right) dx\,dy$$

$$= \oint N_i \left(k_x \frac{\partial T^{(e)}}{\partial x} n_x + k_y \frac{\partial T^{(e)}}{\partial y} n_y\right) dB \qquad (2.2.4)$$

These equations in the matrix form are

$$\iint_{A^{(e)}} (k_x \{N,x\} \lfloor N,x \rfloor + k_y \{N,y\} \lfloor N,y \rfloor)\, dx\, dy\, \{T\}^{(ne)}$$

$$= \oint \{N_i\} (-q)\, dB \qquad (2.2.5)$$

and in short form is

$$[k_t]^{(e)} \{T\}^{(ne)} = \{q\}^{(ne)} \qquad (2.2.6)$$

where, $[k_t]^{(e)}$ is thermal stiffness matrix of the element

$\{T\}^{(ne)}$ is elemental temperature vector

$\{q\}^{(ne)}$ is flux vector along the edges of the element.

2.2.2 With convection

Denoting temperature difference, $T-T_f$, by θ, the governing differential equation (2.1.1) can be written as

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial\theta}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial\theta}{\partial y}\right) = 0 \qquad (2.2.7)$$

and boundary conditions as

$$\theta = \theta_p$$

or

$$k_x \frac{\partial\theta}{\partial x} n_x + k_y \frac{\partial\theta}{\partial y} n_y + q + h\,\theta = 0 \qquad (2.2.8)$$

$\theta^{(e)}$ over the finite element is taken as

$$\theta^{(e)} = \lfloor N \rfloor \{\theta\}^{(ne)} \qquad (2.2.9)$$

Substituting it in equation (2.2.7) one gets

$$R^{(e)} = \frac{\partial}{\partial x}\left(k_x \frac{\partial\theta^{(e)}}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial\theta^{(e)}}{\partial y}\right) \qquad (2.2.10)$$

Minimizing this residue by Galerkin method,

$$\iint_{A^{(e)}} N_i \left[\frac{\partial}{\partial x}\left(k_x \frac{\partial \theta^{(e)}}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial \theta^{(e)}}{\partial y}\right)\right] dx\ dy = 0 \qquad (2.2.11)$$

Doing integration by parts, one gets

$$\iint_{A^{(e)}} \left(N_{i,x}\, k_x \frac{\partial \theta^{(e)}}{\partial x} + N_{i,y}\, k_y \frac{\partial \theta^{(e)}}{\partial y}\right) dx\ dy$$

$$= \oint N_i \left(k_x \frac{\partial \theta^{(e)}}{\partial x} n_x + k_y \frac{\partial \theta^{(e)}}{\partial y} n_y\right) dB$$

$$= \oint N_i \left(-h\, \theta^{(e)} - q\right)\ dB$$

$$= \oint - h\ N_i \lfloor N \rfloor\ dB\ \{\theta\}^{(ne)} - \oint q\ N_i\, dB \qquad (2.2.12)$$

Boundary condition (2.2.8) has been invoked in equation (2.2.12).

These equations in the matrix form are

$$\left[\iint \left(k_x\ \{N,x\}\ \lfloor N,x \rfloor + k_y\ \{N,y\}\ \lfloor N,y \rfloor\right) dx\ dy + \oint h \{N\} \lfloor N \rfloor dB\right] \{\theta\}^{(ne)}$$

$$= \oint - q\ \{N\}\ dB \qquad (2.2.13)$$

and in short form is

$$\left([k_t]^{(e)} + [k_h]^{(e)}\right) \{\theta\}^{(ne)} = \{q\}^{(ne)}$$

or

$$[t]^{(e)}\ \{\theta\}^{(ne)} = \{q\}^{(ne)} \qquad (2.2.14)$$

It may be noted that matrix $[k_h]^{(e)}$ is meaningful only along the edges having convection.

2.3 TFEM equations for heat conduction

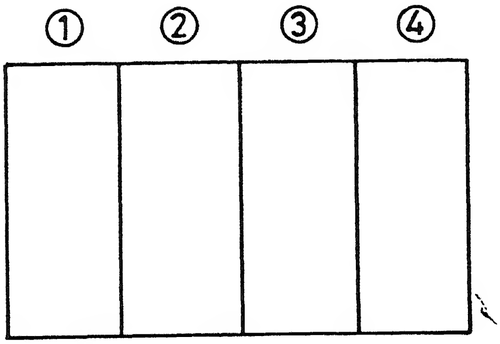


Fig. 2.3.1(a) Rectangular lamina divided into strips

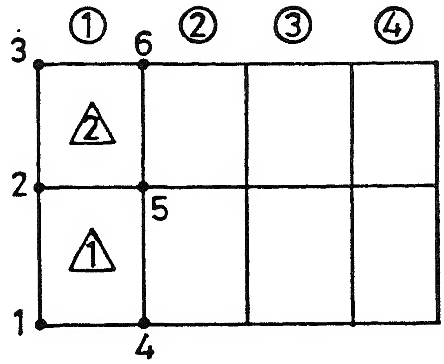


Fig. 2.3.1(b) Strips divided into elements

First the domain is divided into number of finite strips as shown in Fig. 2.3.1(a). Strip directions are so chosen that minimum number of elements are needed in these strips. Strip could have any number of elements, but here each strip is shown to have two elements only, as this reduces the typing work considerably and enables the method to be explained very easily. Here each finite element is taken as four noded rectangular element.

2.3.1 FEM equations of a strip (without convection)

On assembling the elemental stiffness matrices and the flux vectors i.e. equation (2.2.6), one gets FEM equations for the first strip as

$$\begin{bmatrix} k_{11} & k_{12} & \cdots\cdots & k_{16} \\ k_{21} & k_{22} & \cdots\cdots & k_{26} \\ k_{31} & k_{32} & \cdots\cdots & k_{36} \\ k_{41} & k_{42} & \cdots\cdots & k_{46} \\ k_{51} & k_{52} & \cdots\cdots & k_{56} \\ k_{61} & k_{62} & \cdots\cdots & k_{66} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} -ql|_{14} - ql|_{21} \\ -ql|_{21} - ql|_{32} \\ -ql|_{63} - ql|_{32} \\ -ql|_{45} - ql|_{14} \\ -ql|_{45} - ql|_{56} \\ -ql|_{56} - ql|_{63} \end{Bmatrix} \quad (2.3.1)$$

Transfer FEM equations are to be developed from these equations. So the integrals $ql|_{45}$, $ql|_{56}$, $ql|_{63}$ are changed to $ql|_{54}$, $ql|_{65}$, $ql|_{36}$. With this RHS of equation (2.3.1) becomes

$$\begin{Bmatrix} -ql|_{14} - ql|_{21} \\ -ql|_{21} - ql|_{32} \\ -ql|_{32} + ql|_{36} \\ -ql|_{14} + ql|_{54} \\ ql|_{54} + ql|_{65} \\ ql|_{65} + ql|_{36} \end{Bmatrix} \quad (2.3.2)$$

2.3.2 FEM equations of a strip (with convection)

Proceeding as in 2.3.1, the FEM equations for the first strip consisting of elements 1 and 2 (using equations (2.2.14)) are

$$
\begin{bmatrix} t_{11} & t_{12} & \cdots\cdots & t_{16} \\ t_{21} & t_{22} & \cdots\cdots & t_{26} \\ t_{31} & t_{32} & \cdots\cdots & t_{36} \\ \vdots & \vdots & & \vdots \\ t_{61} & t_{62} & \cdots\cdots & t_{66} \end{bmatrix}
\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{Bmatrix}
=
\begin{Bmatrix} -ql|_{14} - ql|_{21} \\ -ql|_{21} - ql|_{32} \\ -ql|_{32} + ql|_{36} \\ -ql|_{14} + ql|_{54} \\ ql|_{54} + ql|_{65} \\ ql|_{65} + ql|_{36} \end{Bmatrix}
\tag{2.3.3}
$$

2.3.3 FEM equations of a strip after applying boundary conditions on its top and bottom

For developing TFEM equations of the strip, boundary conditions on its top and bottom edges should be applied first. Following four cases with different boundary conditions on top and bottom edges are illustrated.

(a) Top and bottom edges insulated.

(b) Top edge with prescribed temperature and bottom edge insulated.

(c) Top and bottom edges with prescribed temperature.

(d) Top edge exposed to fluid (convection) and bottom edge with prescribed temperature.

Case (a) Top and bottom edges insulated

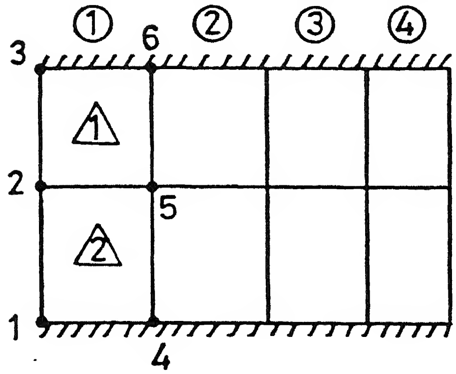


Fig. 2.3.2: Rectangular lamina with top and bottom edges insulated

As edges 14 and 36 are insulated, equations (2.3.1) become

$$\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{16} \\ k_{21} & k_{22} & \cdots & k_{26} \\ \vdots & \vdots & & \vdots \\ k_{61} & k_{62} & \cdots & k_{66} \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{Bmatrix} = \begin{Bmatrix} -ql|_{21} \\ -ql|_{32} - ql|_{21} \\ -ql|_{32} \\ ql|_{54} \\ ql|_{65} + ql|_{54} \\ ql|_{65} \end{Bmatrix} \qquad (2.3.4)$$

Case(b) Temperature prescribed on top edge and insulated on the bottom edge

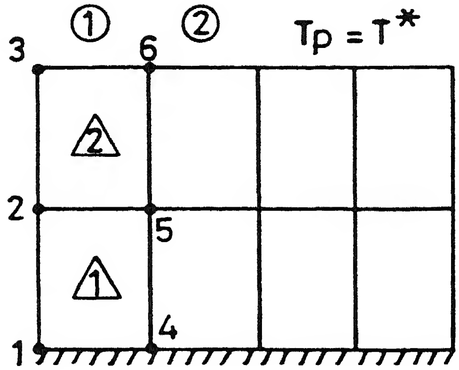


Fig.2.3.3 Rectangular lamina with bottom edge insulated and top edge having prescribed temperature

Applying boundary conditions of edge 36 (i.e. $T_3 = T_3^*$ and $T_6 = T_6^*$) and edge 14 (i.e. $ql|_{14} = 0$), FEM equations for this strip become

$$\underset{6\times5}{\begin{bmatrix} k_{11} & k_{12} & k_{14} & k_{15} & (k_{13}T_3^*+k_{16}T_6^*) \\ k_{21} & k_{22} & k_{24} & k_{25} & (k_{23}T_3^*+k_{26}T_6^*) \\ k_{31} & k_{32} & k_{34} & k_{35} & (k_{33}T_3^*+k_{36}T_6^*) \\ k_{41} & k_{42} & k_{44} & k_{45} & (k_{43}T_6^*+k_{46}T_6^*) \\ k_{51} & k_{52} & k_{54} & k_{55} & (k_{53}T_3^*+k_{56}T_6^*) \\ k_{61} & k_{62} & k_{64} & k_{65} & (k_{63}T_3^*+k_{66}T_6^*) \end{bmatrix}} \underset{5\times1}{\left\{\begin{matrix} T_1 \\ T_2 \\ T_4 \\ T_5 \\ 1 \end{matrix}\right\}} = \underset{6\times1}{\left\{\begin{matrix} -ql|_{21} \\ -ql|_{21} - ql|_{32} \\ -ql|_{32} + ql|_{36} \\ ql|_{54} \\ ql|_{65} + ql|_{54} \\ ql|_{65} + ql|_{36} \end{matrix}\right\}}$$

(2.3.5)

It will be seen later that TFEM equations for this strip can be obtained from first, second, fourth and fifth equation of

(2.3.5) and third or sixth equation can later be used to find flux along edge 36. Thus writing the above mentioned four equations of (2.3.5)

$$\begin{bmatrix} k_{11} & k_{12} & k_{14} & k_{15} & (k_{13}T_3^*+k_{16}T_6^*) \\ k_{21} & k_{22} & k_{24} & k_{25} & (k_{23}T_3^*+k_{26}T_6^*) \\ k_{41} & k_{42} & k_{44} & k_{45} & (k_{43}T_3^*+k_{46}T_6^*) \\ k_{51} & k_{52} & k_{54} & k_{55} & (k_{53}T_3^*+k_{56}T_6^*) \\ & & & & \end{bmatrix} \begin{Bmatrix} T_1 \\ T_2 \\ T_4 \\ T_5 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -ql|_{21} \\ -ql|_{21} - ql|_{32} \\ ql|_{54} \\ ql|_{65} + ql|_{54} \\ \end{Bmatrix} \qquad (2.3.6)$$

Case (c) Temperature prescribed on bottom and top edges

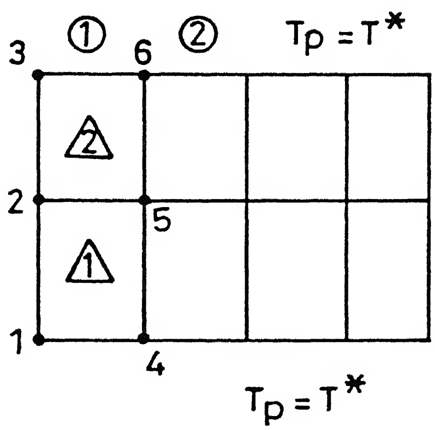


Fig. 2.3.4 Rectangular lamina with temperature prescribed on opposite edges

Applying boundatry conditions of edge 36 (i.e. $T_3 = T_3^*$ and $T_6 = T_6^*$) and edge 14 (i.e. $T_1 = T_1^*$ and $T_4 = T_4^*$), FEM equations (2.3.1) and (2.3.2) for this strip become

$$\begin{bmatrix} k_{12} & k_{15} & (k_{11}T_1^*+k_{13}T_3^*+k_{14}T_4^*+k_{16}T_6^*) \\ k_{22} & k_{25} & (k_{21}T_1^*+k_{23}T_3^*+k_{24}T_4^*+k_{26}T_6^*) \\ k_{32} & k_{35} & (k_{31}T_1^*+k_{33}T_3^*+k_{34}T_4^*+k_{36}T_6^*) \\ k_{42} & k_{45} & (k_{41}T_1^*+k_{43}T_3^*+k_{44}T_4^*+k_{46}T_6^*) \\ k_{52} & k_{55} & (k_{51}T_1^*+k_{53}T_3^*+k_{54}T_4^*+k_{56}T_6^*) \\ k_{62} & k_{65} & (k_{61}T_1^*+k_{63}T_3^*+k_{64}T_4^*+k_{66}T_6^*) \end{bmatrix} \begin{Bmatrix} T_2 \\ T_5 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -ql|_{14}-ql|_{21} \\ -ql|_{21}-ql|_{32} \\ -ql|_{32}+ql|_{36} \\ -ql|_{14}+ql|_{54} \\ ql|_{54}+ql|_{65} \\ ql|_{65}+ql|_{36} \end{Bmatrix}$$

(2.3.7)

It will be seen later that TFEM equations for this strip can be obtained from second and fifth equation of (2.3.7) and remaining equations can later be used to find flux across the edges 14 and 36. Thus writing the above mentioned two equations of (2.3.7)

$$\begin{bmatrix} k_{22} & k_{25} & (k_{21}T_1^*+k_{23}T_3^*+k_{24}T_4^*+k_{26}T_6^*) \\ k_{52} & k_{55} & (k_{51}T_1^*+k_{53}T_3^*+k_{54}T_4^*+k_{56}T_6^*) \end{bmatrix} \begin{Bmatrix} T_2 \\ T_5 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -ql|_{21}-ql|_{32} \\ ql|_{54}+ql|_{65} \end{Bmatrix}$$

(2.3.8)

Case(d) Fluid on top edge and prescribed temperature on the bottom edge

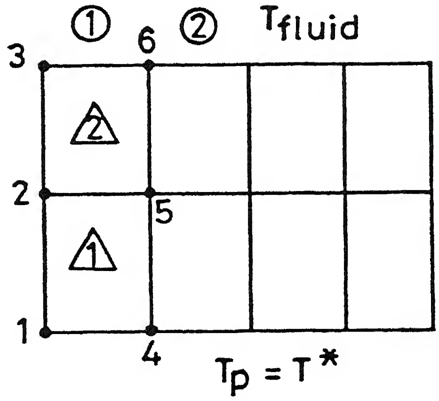


Fig.2.3.5 Rectangular lamina with fluid on top edge and prescribed temperature on the bottome edge

Applying the boundary conditions of edge 14 (i.e. $\theta^*_1 = T^*_1 - T_f$ and $\theta^*_4 = T^*_4 - T_f$), and noting the convection on edge 36 FEM equations (2.3.3) for this strip become

$$\begin{bmatrix} t_{12} & t_{13} & t_{15} & t_{16} & (t_{11}\theta^*_1 + t_{14}\theta^*_4) \\ t_{22} & t_{23} & t_{25} & t_{26} & (t_{21}\theta^*_1 + t_{24}\theta^*_4) \\ t_{32} & t_{33} & t_{35} & t_{36} & (t_{31}\theta^*_1 + t_{34}\theta^*_4) \\ t_{42} & t_{43} & t_{45} & t_{46} & (t_{41}\theta^*_1 + t_{44}\theta^*_4) \\ t_{52} & t_{53} & t_{55} & t_{56} & (t_{51}\theta^*_1 + t_{54}\theta^*_4) \\ t_{62} & t_{63} & t_{65} & t_{66} & (t_{61}\theta^*_1 + t_{64}\theta^*_4) \end{bmatrix} \left\{ \begin{matrix} \theta_2 \\ \theta_3 \\ \theta_5 \\ \theta_6 \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} -ql|_{14} - ql|_{21} \\ -ql|_{21} - ql|_{32} \\ -ql|_{32} \\ -ql|_{14} + ql|_{54} \\ ql|_{54} + ql|_{65} \\ ql|_{65} \end{matrix} \right\} \qquad (2.3.9)$$

It will be seen later that TFEM equations for this strip can be obtained from second, third, fifth and sixth equation of (2.3.9), and first or fourth equation can later be used to find

flux along edge 14. Thus writing the above mentioned four equations of (2.3.9)

$$\begin{bmatrix} t_{22} & t_{23} & t_{25} & t_{26} & (t_{21}\theta_1^*+t_{24}\theta_4^*) \\ t_{32} & t_{33} & t_{35} & t_{36} & (t_{31}\theta_1^*+t_{34}\theta_4^*) \\ t_{52} & t_{53} & t_{55} & t_{56} & (t_{51}\theta_1^*+t_{54}\theta_4^*) \\ t_{62} & t_{63} & t_{65} & t_{66} & (t_{61}\theta_1^*+t_{64}\theta_4^*) \end{bmatrix} \left\{ \begin{matrix} \theta_2 \\ \theta_3 \\ \theta_5 \\ \theta_6 \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} -ql|_{21} - ql|_{32} \\ -ql|_{32} \\ ql|_{54} + ql|_{65} \\ ql|_{65} \end{matrix} \right\} \tag{2.3.10}$$

2.3.4 Transfer finite element equations

The transfer FEM equations are now developed for all cases discussed in 2.3.3. It involves partitioning and rearranging of matrices of the strip.

2.3.4.1 TFEM equations for case (a)

Denoting the flux $-ql|_{32} - ql|_{21}$ by $-ql|_{31}$ and $ql|_{65} + ql|_{54}$ by $ql|_{64}$, the FEM equations (2.3.4) are rewritten and partitioned as indicated.

$$\left[\begin{array}{ccc:ccc} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ \hdashline k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{array}\right] \left\{\begin{array}{c} T_1 \\ T_2 \\ T_3 \\ \hdashline T_4 \\ T_5 \\ T_6 \end{array}\right\} = \left\{\begin{array}{c} -ql|_{21} \\ -ql|_{31} \\ -ql|_{32} \\ \hdashline ql|_{54} \\ ql|_{64} \\ ql|_{65} \end{array}\right\} \tag{2.3.11}$$

where T_1, T_2, T_3 are temperatures and $-ql|_{21}$, $-ql|_{31}$, $-ql|_{32}$ are

heat fluxes for the left side of this strip; and T_4, T_5, T_6 are temperatures and $ql|_{54}$, $ql|_{64}$ and $ql|_{65}$ are heat fluxes for the right side of this strip. Equation (2.3.11) are written in short form as

$$\begin{bmatrix} [A] & [B] \\ [C] & [D] \end{bmatrix} \begin{Bmatrix} \{T\}_1 \\ \{T\}_2 \end{Bmatrix} = \begin{Bmatrix} -\{q\}_1 \\ \{q\}_2 \end{Bmatrix} \qquad (2.3.12)$$

Subscript 1 outside the matrix denotes left side of the strip and 2 denotes right side of the strip. Equations (2.3.12) can be written in the following form,

$$[A]\ \{T\}_1 + [B]\ \{T\}_2 = -\{q\}_1 \qquad (2.3.13)$$

$$[C]\ \{T\}_1 + [D]\ \{T\}_2 = \{q\}_2 \qquad (2.3.14)$$

Equation (2.3.13) gives,

$$\{T\}_2 = [B]^{-1}\left[-\{q\}_1 - [A]\ \{T\}_1\right] \qquad (2.3.15)$$

Suubstituting this value of $\{T\}_2$ in equation (2.3.14) one gets,

$$\left([C] - [D]\ [B]^{-1}[A]\right)\ \{T\}_1 - [D]\ [B]^{-1}\{q\}_1 = \{q\}_2 \qquad (2.3.16)$$

Combining equations (2.3.15) and (2.3.16) in matrix form, one gets,

$$\begin{Bmatrix} \{T\}_2 \\ \{q\}_2 \end{Bmatrix} = \begin{bmatrix} -[B]^{-1}\ [A] & -[B]^{-1} \\ ([C]-[D][B]^{-1}[A]) & -[D][B]^{-1} \end{bmatrix} \begin{Bmatrix} \{T\}_1 \\ \{q\}_1 \end{Bmatrix}$$

(2.3.17)

or

$$\begin{Bmatrix} \{T\}_2 \\ \{q\}_2 \end{Bmatrix} = [TFM]_1 \begin{Bmatrix} \{T\}_1 \\ \{q\}_1 \end{Bmatrix} \qquad (2.3.18)$$

These are TFEM equations. Knowing temperatures and fluxes on one side of the strip, one can obtain temperatures and fluxes on the other side.

2.3.4.2 Transfer FEM equations for cases (b), (c) and (d)

FEM equations for all these cases are of same nature. So the TFEM equations here are developed for case (b) and their development for other cases will be obvious.

Rewriting equation (2.3.6), and denoting $-ql|_{32} - ql|_{21}$ by $-ql|_{31}$ and $ql|_{65} + ql|_{54}$ by $ql|_{64}$ one gets partitioned as indicated.

$$\left[\begin{array}{cc:cc:c} k_{11} & k_{12} & k_{14} & k_{15} & (k_{13}T_3^* + k_{16}T_6^*) \\ k_{21} & k_{22} & k_{24} & k_{25} & (k_{23}T_3^* + k_{26}T_6^*) \\ \hdashline k_{41} & k_{42} & k_{44} & k_{45} & (k_{43}T_3^* + k_{46}T_6^*) \\ k_{51} & k_{52} & k_{54} & k_{55} & (k_{53}T_3^* + k_{56}T_6^*) \end{array}\right] \left\{\begin{array}{c} T_1 \\ T_2 \\ \hdashline T_4 \\ T_5 \\ 1 \end{array}\right\} = \left\{\begin{array}{c} -ql|_{21} \\ -ql|_{31} \\ \hdashline ql|_{54} \\ ql|_{64} \end{array}\right\} \qquad (2.3.19)$$

T_1, T_2 are temperatures and $-ql|_{21}$, $-ql|_{31}$ are heat fluxes for the left side of the strip; and T_4, T_5 are temperatures and $ql|_{54}$ and $ql|_{64}$ are heat fluxes for the right side of the strip. Equations (2.3.19) are partitioned as indicated. Writing these in short form as,

$$\begin{bmatrix} [A] & [B] & \{C\} \\ [D] & [E] & \{F\} \end{bmatrix} \left\{\begin{array}{c} \{T\}_1 \\ \{T\}_2 \\ 1 \end{array}\right\} = \left\{\begin{array}{c} -\{q\}_1 \\ \{q\}_2 \end{array}\right\} \qquad (2.3.20)$$

Subscripts 1 and 2 refer to left and right sides of the

strip as before.

Equation (2.3.20) can be written as:

$$[A]\{T\}_1 + [B]\{T\}_2 + \{C\} = -\{q\}_1 \qquad (2.3.21)$$

and

$$[D]\{T\}_1 + [E]\{T\}_2 + \{F\} = \{q\}_2 \qquad (2.3.22)$$

From equation (2.3.21) one gets

$$\{T\}_2 = [B]^{-1}\Big[-\{q\}_1 - [A]\{T\}_1 - \{C\}\Big] \qquad (2.3.23)$$

Substituting this value of $\{T\}_2$ in equation (2.3.22), one gets,

$$\{q\}_2 = \Big([D]-[E][B]^{-1}[A]\Big)\{T\}_1+\Big(-[E][B]^{-1}\Big)\{q\}_1 + \Big(\{F\} - [E][B]^{-1}\{C\}\Big)$$

(2.3.24)

Combining equations (2.3.23) and (2.3.24), one gets

$$\begin{Bmatrix} \{T\}_2 \\ \\ \{q\}_2 \end{Bmatrix} = \begin{bmatrix} -[B]^{-1}[A] & -[B]^{-1} & -[B]^{-1}\{C\} \\ \\ ([D]-[E][B]^{-1}[A]) & -[E][B]^{-1} & (\{F\}-[E][B]^{-1}\{C\}) \end{bmatrix} \begin{Bmatrix} \{T\}_1 \\ \\ \{q\}_1 \\ 1 \end{Bmatrix}$$

(2.3.25)

In this equations, state vectors on left and right side are not of the same type. To make these of same type, these equations are modified as follows.

On combining equations (2.3.25) and (2.3.26), one gets

$$\begin{Bmatrix} \{T\}_2 \\ \\ \{q\}_2 \\ 1 \end{Bmatrix} = \begin{bmatrix} -[B]^{-1}[A] & -[B]^{-1} & -[B]^{-1}\{C\} \\ \\ ([D]-[E][B]^{-1}[A]) & -[E][B]^{-1} & (\{F\}-[E][B]^{-1}\{C\}) \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & 1 \end{bmatrix} \begin{Bmatrix} \{T\}_1 \\ \\ \{q\}_1 \\ 1 \end{Bmatrix} \tag{2.3.26}$$

or

$$\begin{Bmatrix} \{T\}_2 \\ \\ \{q\}_2 \\ 1 \end{Bmatrix} = [TFM]_1 \begin{Bmatrix} \{T\}_1 \\ \\ \{q\}_1 \\ 1 \end{Bmatrix} \tag{2.3.27}$$

These are TFEM equations, one can obtain temperatures and fluxes on the right side of the strip, knowing these values on the left hand side.

2.4 TFEM equations for conduction with eight noded elements

As in section 2.3, each strip is supposed to have two elements. But here finite elements are eight noded isoparametric elements (Fig. 2.4.1). The approach of section 2.3 cannot be used directly to develop TFEM equations for this strip because of the intermediate nodes like 11, 12 and 13. These nodes are condensed out in order to develop the transfer finite element equations.

For explaining this method, TFEM equations are developed for two cases. In the first case there is convection on one edge and opposite edge insulated. And in the second case there is convection on one edge and opposite edge has prescribed temperature. Extension to other boundary conditions on the opposite edges will be obvious.

Case (a) : Convection on one edge and insulation on the other edge

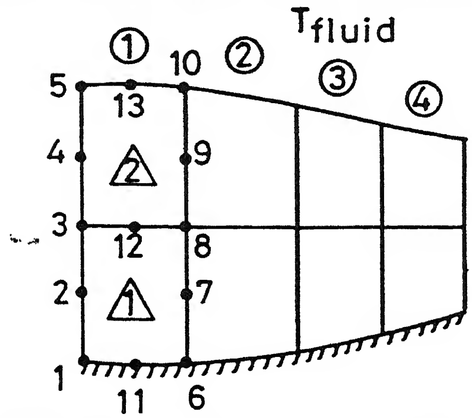


Fig. 2.4.1 : Plane lamina with convection on one edge and insulation on the other edge

Writing FEM equations for the strip using equation (2.2.14).

$$
\begin{bmatrix}
t_{11} & t_{12} & \cdots\cdots & t_{1,13} \\
t_{21} & t_{33} & \cdots\cdots & t_{2,13} \\
\vdots & & & \vdots \\
t_{13,1} & t_{13,2} & & t_{13,13}
\end{bmatrix}
\begin{Bmatrix}
\theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \\ \theta_7 \\ \theta_8 \\ \theta_9 \\ \theta_{10} \\ \theta_{11} \\ \theta_{12} \\ \theta_{13}
\end{Bmatrix}
=
\begin{Bmatrix}
-ql_{31} - ql_{16} \\
-ql|_{31} \\
-ql|_{53} - ql|_{31} \\
-ql|_{53} \\
-ql|_{53} \\
-ql|_{68} - ql|_{16} \\
-ql|_{68} \\
-ql|_{68} - ql|_{8,10} \\
-ql|_{8,10} \\
-ql|_{8,10} \\
-ql|_{16} \\
0 \\
0
\end{Bmatrix}
\tag{2.4.1}
$$

To obtain the TFEM equations the integrals $ql|_{68}$, $ql|_{8,10}$ and $ql|_{10,5}$ will be changed to $ql|_{86}$, $ql|_{10,8}$ and $ql|_{5,10}$. Also the boundary conditions on the top edge 5-13-10 (convection) and

bottom edge 1-11-6 ($ql|_{1,11} = ql|_{11,6} = 0$) are applied. Thus equations (2.4.1) become

$$
\left[\begin{array}{cccc|ccc|ccc}
t_{11} & t_{12} & \cdots & t_{15} & t_{16} & \cdots & t_{1,10} & t_{1,11} & \cdots & t_{1,13} \\
t_{21} & t_{22} & \cdots & t_{25} & t_{26} & \cdots & t_{2,10} & t_{2,11} & \cdots & t_{2,13} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
t_{51} & t_{52} & \cdots & t_{55} & t_{56} & \cdots & t_{5,10} & t_{5,11} & \cdots & t_{5,13} \\
t_{61} & t_{62} & \cdots & t_{65} & t_{66} & \cdots & t_{6,10} & t_{6,11} & \cdots & t_{6,13} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
t_{10,1} & t_{10,2} & \cdots & t_{10,5} & t_{10,6} & \cdots & t_{10,10} & t_{10,11} & \cdots & t_{10,13} \\
t_{11,1} & t_{11,2} & \cdots & t_{11,5} & t_{11,6} & \cdots & t_{11,10} & t_{11,11} & \cdots & t_{11,13} \\
\vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & & \vdots \\
t_{13,1} & t_{13,2} & \cdots & t_{13,5} & t_{13,6} & \cdots & t_{13,10} & t_{13,11} & \cdots & t_{13,13}
\end{array}\right]
\left\{\begin{array}{c}
\theta_1 \\ \theta_2 \\ \vdots \\ \theta_5 \\ \theta_6 \\ \vdots \\ \theta_{10} \\ \theta_{11} \\ \vdots \\ \theta_{13}
\end{array}\right\}
=
\left\{\begin{array}{c}
-ql|_{31} \\
-ql|_{31} \\
-ql|_{53} - ql|_{31} \\
-ql|_{53} \\
-ql|_{53} \\
\cdots\cdots\cdots \\
ql|_{86} \\
ql|_{86} \\
ql|_{86} + ql|_{10,8} \\
ql|_{10,8} \\
ql|_{10,8} \\
\cdots\cdots\cdots \\
0 \\
0 \\
0
\end{array}\right\}
\qquad (2.4.2)
$$

Temperatures at the intermediate nodes of the strip are not required in the TFEM equation of the strip. They can be condensed from equations (2.4.2) [19].

Equations (2.4.2) are partitioned as indicated and are rewritten as

$$\begin{bmatrix} [A1] & [A2] & [A3] \\ [A4] & [A5] & [A6] \\ [A7] & [A8] & [A9] \end{bmatrix} \begin{Bmatrix} \{T\}_1 \\ \{T\}_2 \\ \{T\}_3 \end{Bmatrix} = \begin{Bmatrix} -\{q\}_1 \\ \{q\}_2 \\ \{0\} \end{Bmatrix} \qquad (2.4.3)$$

Third equation of equations (2.4.3) gives,

$$\{T\}_3 = [A9]^{-1} \left[-[A7]\{T\}_1 - [A8]\,\{T\}_2 \right] \qquad (2.4.4)$$

Substituting this equation in the first two equations of equations (2.4.3), one gets,

$$\begin{bmatrix} -([A1]-[A3][A9]^{-1}[A7]) & -([A2]-[A3][A9]^{-1}[A8]) \\ ([A4]-[A6][A9]^{-1}[A7]) & ([A5]-[A6][A9]^{-1}[A8]) \end{bmatrix} \begin{Bmatrix} \{T\}_1 \\ \{T\}_2 \end{Bmatrix} = \begin{Bmatrix} \{q\}_1 \\ \{q\}_2 \end{Bmatrix}$$

(2.4.5)

or

$$\begin{bmatrix} [B1] & [B2] \\ [B3] & [B4] \end{bmatrix} \begin{Bmatrix} \{T\}_1 \\ \{T\}_2 \end{Bmatrix} = \begin{Bmatrix} \{q\}_1 \\ \{q\}_2 \end{Bmatrix} \qquad (2.4.6)$$

These equations are rearranged as

$$\begin{Bmatrix} \{T\}_2 \\ \{q\}_2 \end{Bmatrix} = \begin{bmatrix} -[B2]^{-1}[B1] & [B2]^{-1} \\ [B3]-[B4][B2]^{-1}[B1] & [B4][B2]^{-1} \end{bmatrix} \begin{Bmatrix} \{T\}_1 \\ \{q\}_1 \end{Bmatrix}$$

or

$$\begin{Bmatrix} \{T\}_2 \\ \{q\}_2 \end{Bmatrix} = [TFM]_1 \begin{Bmatrix} \{T\}_1 \\ \{q\}_1 \end{Bmatrix} \qquad (2.4.7)$$

Case (b) : Convection on one edge and temperature prescribed on the other edge

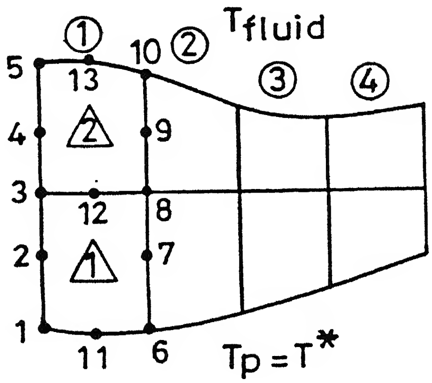


Fig.2.4.2 : Plane lamina with convection on one edge and temperature prescribed on other edge

FEM equations for the strip 1 are the equations (2.4.1). To convert these equations into TFEM equations, the boundary conditions on bottom edge 1-11-6 (i.e. $\theta_1^* = T_1^* - T_f$, $\theta_{11}^* = T_{11}^* - T_f$ and $\theta_6^* = T_6^* - T_f$) are applied and applying convection boundary condition on the top edge 5-13-10, one gets

$$
\begin{bmatrix}
t_{12} \cdots\cdots t_{1,13} & (t_{11}\,\theta_1^* + t_{16}\theta_6^* + t_{1,11}\theta_{11}^*) \\
t_{22} \cdots\cdots t_{2,13} & (t_{21}\,\theta_1^* + t_{26}\theta_6^* + t_{2,11}\theta_{11}^*) \\
t_{32} \cdots\cdots t_{3,13} & (t_{31}\,\theta_1^* + t_{36}\theta_6^* + t_{3,11}\theta_{11}^*) \\
\vdots \qquad\quad \vdots & \vdots \\
t_{13,1} \cdots\cdots t_{13,13} & (t_{13,1}\,\theta_1^* + t_{13,6}\theta_6^* + t_{13,11}\theta_{11}^*)
\end{bmatrix}
\begin{Bmatrix}
\theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_7 \\ \theta_8 \\ \theta_9 \\ \theta_{10} \\ \theta_{12} \\ \theta_{13} \\ 1
\end{Bmatrix}
=
\begin{Bmatrix}
-ql|_{31} - ql|_{16} \\
-ql|_{31} \\
-ql|_{53} - ql|_{31} \\
-ql|_{53} \\
-ql|_{53} \\
-ql|_{68} - ql|_{16} \\
-ql|_{68} \\
-ql|_{68} - ql|_{8,10} \\
-ql|_{8,10} \\
-ql|_{8,10} \\
-ql|_{16} \\
0 \\
0
\end{Bmatrix}
\qquad (2.4.8)
$$

Transfer matrix for this strip can be obtained from second, third, fourth, fifth, seventh, eighth, ninth, tenth, twelfth and

thirteenth rows. Any one of the dropped out rows can later be used to find the flux along edge 16. Thus writing the above mentioned ten equations

$$
\underset{10\times 11}{\begin{bmatrix}
t_{22}\cdots\cdots t_{2,13} & (t_{21}\theta_1^* + t_{26}\theta_6^* + t_{2,11}\theta_{11}^*) \\
t_{32}\cdots\cdots t_{3,13} & (t_{31}\theta_1^* + t_{36}\theta_6^* + t_{3,11}\theta_{11}^*) \\
t_{42}\cdots\cdots t_{4,13} & (t_{41}\theta_1^* + t_{46}\theta_6^* + t_{4,11}\theta_{11}^*) \\
t_{52}\cdots\cdots t_{5,13} & (t_{51}\theta_1^* + t_{56}\theta_6^* + t_{5,11}\theta_{11}^*) \\
t_{72}\cdots\cdots t_{7,13} & (t_{71}\theta_1^* + t_{76}\theta_6^* + t_{7,11}\theta_{11}^*) \\
t_{82}\cdots\cdots t_{8,13} & (t_{81}\theta_1^* + t_{86}\theta_6^* + t_{8,11}\theta_{11}^*) \\
t_{92}\cdots\cdots t_{9,13} & (t_{91}\theta_1^* + t_{96}\theta_6^* + t_{9,11}\theta_{11}^*) \\
t_{10,2}\cdots\cdots t_{10,13} & (t_{10,1}\theta_1^* + t_{10,6}\theta_6^* + t_{10,11}\theta_{11}^*) \\
t_{12,2}\cdots\cdots t_{12,13} & (t_{12,1}\theta_1^* + t_{12,6}\theta_6^* + t_{12,11}\theta_{11}^*) \\
t_{13,2}\cdots\cdots t_{13,13} & (t_{13,1}\theta_1^* + t_{13,6}\theta_6^* + t_{13,11}\theta_{11}^*)
\end{bmatrix}}
\underset{11\times 1}{\begin{bmatrix}
\theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \cdots\cdots \\ \theta_7 \\ \theta_8 \\ \theta_9 \\ \theta_{10} \\ \cdots\cdots \\ \theta_{12} \\ \theta_{13} \\ \cdots\cdots \\ 1
\end{bmatrix}}
$$

$$
= \underset{10\times 1}{\begin{bmatrix}
-ql|_{31} \\
-ql|_{53} - ql|_{31} \\
-ql|_{53} \\
-ql|_{53} \\
\cdots\cdots\cdots\cdots \\
ql|_{86} \\
ql|_{86} + ql|_{10,8} \\
ql|_{10,8} \\
ql|_{10,8} \\
\cdots\cdots\cdots\cdots \\
0 \\
0
\end{bmatrix}} \qquad (2.4.9)
$$

Temperatures at the intermediate nodes of the strip are not required in the TFEM equations of the strip. These can be condensed out, [19].

Equations (2.4.9) are partitioned and rewritten as,

$$\begin{bmatrix} \underset{4\times4}{[A1]} & \underset{4\times4}{[A2]} & \underset{4\times2}{[A3]} & \underset{4\times1}{\{X1\}} \\ \\ \underset{4\times4}{[A4]} & \underset{4\times4}{[A5]} & \underset{4\times2}{[A6]} & \underset{4\times1}{\{X2\}} \\ \\ \underset{2\times4}{[A7]} & \underset{2\times4}{[A8]} & \underset{2\times2}{[A9]} & \underset{2\times1}{\{X3\}} \end{bmatrix} \begin{Bmatrix} \{\theta\}_1 \\ \\ \{\theta\}_2 \\ \\ \{\theta\}_3 \\ 1 \end{Bmatrix} = \begin{Bmatrix} -\{q\}_1 \\ \\ \{q\}_2 \\ \\ \{0\} \end{Bmatrix}$$

(2.4.10)

Third equation of equations (2.4.10) gives,

$$\{\theta\}_3 = [A9]^{-1} \left[-[A7]\,\{\theta\}_1 - [A8]\,\{\theta\}_2 - \{X3\} \right] \qquad (2.4.11)$$

Substituting this equation in the first two equations of equations (2.4.10), one gets

$$\begin{bmatrix} -([A1]-[A3][A9]^{-1}[A7]) & -\,([A2]-[A3][A9]^{-1}[A8]) & -\,(\{X1\}-[A3][A9]^{-1}\{X3\}) \\ \\ ([A4]-[A6][A9]^{-1}[A7]) & ([A5]-[A6][A9]^{-1}[A8]) & (\{X2\}-[A6][A9]^{-1}\{X3\}) \end{bmatrix}$$

$$\begin{Bmatrix} \{\theta\}_1 \\ \{\theta\}_2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} \{q\}_1 \\ \\ \{q\}_2 \end{Bmatrix} \qquad (2.4.12)$$

or

$$\begin{bmatrix} [B1] & [B2] & \{B3\} \\ \\ [B4] & [B5] & \{B6\} \end{bmatrix} \begin{Bmatrix} \{\theta\}_1 \\ \\ \{\theta\}_2 \\ 1 \end{Bmatrix} = \begin{Bmatrix} \{q\}_1 \\ \\ \{q\}_2 \end{Bmatrix} \qquad (2.4.13)$$

These equations are rearranged as,

$$\begin{Bmatrix} \{\theta\}_2 \\ \{q\}_2 \end{Bmatrix} = \begin{bmatrix} -[B2]^{-1}[B1] & [B2]^{-1} & -[B2]^{-1}\{B3\} \\ ([B4]-[B5][B2]^{-1}[B1]) & [B5][B2]^{-1} & ([B6]-[B5][B2]^{-1}[B \end{bmatrix} \begin{Bmatrix} \{\theta\}_1 \\ \{q\}_1 \\ 1 \end{Bmatrix} \qquad (2.4.14)$$

Making state vectors (on left and right sides of the strip) of the same type, these equations get modified as

$$\begin{Bmatrix} \{\theta\}_2 \\ \{q\}_2 \\ 1 \end{Bmatrix} = \begin{bmatrix} -[B2]^{-1}[B1] & [B2]^{-1} & [B2]^{-1}\{B3\} \\ ([B4]-[B5][B2]^{-1}[B1]) & [B5][B2]^{-1} & ([B6]-[B5][B2]^{-1}\{B3\}) \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & 1 \end{bmatrix} \begin{Bmatrix} \{\theta\}_1 \\ \{q\}_1 \\ 1 \end{Bmatrix} \qquad (2.4.15)$$

or

$$\begin{Bmatrix} \{\theta\}_2 \\ \{q\}_2 \\ 1 \end{Bmatrix} = [TFM]_1 \begin{Bmatrix} \{\theta\}_1 \\ \{q\}_1 \\ 1 \end{Bmatrix} \qquad (2.4.16)$$

2.5 TFEM equations for stress analysis

In this section TFEM equations are developed first for a domain with four noded finite elements and then with eight noded finite elements.

Case(a): Domain with four noded finite elements

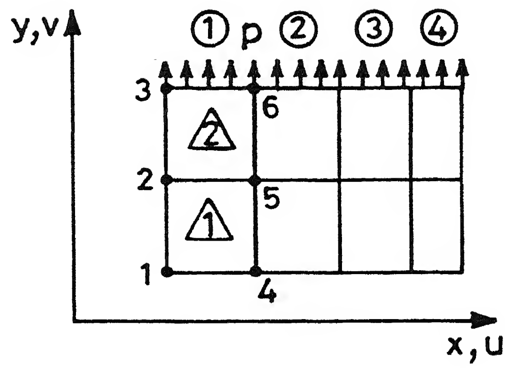


Fig.2.5.1 : Rectangular lamina with pressure p applied on its upper edge

As before, lamina is first divided into number of strips. Strip can be divided into any number of finite elements. For reducing typing work and easy understanding, each strip is supposed to have two elements only.

FEM equations for a four noded element are [21]

$$[K]^{(e)} \{d\}^{(ne)} = \{P\}^{(ne)} + \{F\}^{(ne)} \qquad (2.5.1)$$

where,

$[K]^{(e)}$ is the elemental stiffness matrix

$\{P\}^{(ne)}$ is the elemental load matrix

$\{F\}^{(ne)}$ is the force vector along the edges of the element

FEM equations for the first strip with two elements become

$$
\begin{bmatrix}
K_{11} & K_{12} & \cdots & K_{1,12} \\
K_{21} & K_{22} & \cdots & K_{2,12} \\
K_{31} & K_{32} & \cdots & K_{3,12} \\
K_{41} & K_{42} & \cdots & K_{4,12} \\
\vdots & & & \vdots \\
K_{12,1} & \cdots & \cdots & K_{12,12}
\end{bmatrix}
\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{Bmatrix}
+
\begin{Bmatrix}
Fx_{14} + Fx_{21} \\
Fy_{14} + Fy_{21} \\
Fx_{32} + Fx_{21} \\
Fy_{32} + Fy_{21} \\
Fx_{32} + Fx_{63} \\
Fy_{32} + Fy_{63} \\
Fx_{14} + Fx_{45} \\
Fy_{14} + Fy_{45} \\
Fx_{45} + Fx_{56} \\
Fy_{45} + Fy_{56} \\
Fx_{56} + Fx_{63} \\
Fy_{56} + Fy_{63}
\end{Bmatrix}
\tag{2.5.2}
$$

Since TFEM equations are to be developed from equation (2.5.2), the edge forces Fx_{45}, Fy_{45}, Fx_{56}, Fy_{56} are changed to $-Fx_{54}$, $-Fy_{54}$, $-Fx_{65}$, $-Fy_{65}$. Noting the boundary conditions on the bottom edge 14 ($Fx_{14} = Fy_{14} = 0$) and on top edge 36 ($Fx_{36} = 0$; Fy_{36} by Py_3 and Py_6), equations (2.5.2)

$$
\left[\begin{array}{cccc:ccc:c}
K_{11} & K_{12} & \cdots & K_{15} & K_{16} & K_{17} \cdots & K_{1,12} & 0 \\
K_{21} & K_{22} & \cdots & K_{25} & K_{26} & K_{27} \cdots & K_{2,12} & 0 \\
\vdots & \vdots & & & & & & \vdots \\
K_{51} & K_{52} & \cdots & K_{55} & K_{56} & K_{57} \cdots & K_{5,12} & 0 \\
K_{61} & K_{62} & \cdots & K_{65} & K_{66} & K_{67} \cdots & K_{6,12} & -Py_{3} \\
\hdashline
K_{71} & K_{72} & \cdots & K_{75} & K_{76} & K_{77} \cdots & K_{7,12} & 0 \\
K_{81} & K_{82} & \cdots & K_{85} & K_{86} & K_{87} \cdots & K_{8,12} & 0 \\
\vdots & \vdots & & & & & & \vdots \\
K_{11,1} & K_{11,2} & \cdots & K_{11,5} & K_{11,6} & K_{12,7} \cdots & K_{12,12} & 0 \\
K_{12,1} & K_{12,2} & \cdots & K_{12,5} & K_{12,6} & K_{12,7} \cdots & K_{12,12} & -Py_{6}
\end{array}\right]
\left\{\begin{array}{c}
u_1 \\ v_1 \\ \vdots \\ u_3 \\ v_3 \\ \hdashline u_4 \\ v_4 \\ \vdots \\ u_6 \\ v_6 \\ 1
\end{array}\right\}
=
\left\{\begin{array}{c}
Fx_{21} \\ Fy_{21} \\ Fx_{32} + Fx_{21} \\ Fy_{32} + Fy_{21} \\ Fx_{32} \\ Fy_{32} \\ \hdashline -Fx_{54} \\ -Fy_{54} \\ -Fx_{54} - Fx_{65} \\ -Fy_{54} - Fy_{65} \\ -Fx_{65} \\ -Fy_{65}
\end{array}\right\}
\qquad (2.5.3)
$$

Partitioning equation (2.5.3) as indicated, one gets,

$$
\begin{bmatrix} [A1] & [A2] & \{X1\} \\ [A3] & [A4] & \{X2\} \end{bmatrix}
\left\{\begin{array}{c} \{u\}_1 \\ \{u\}_2 \\ 1 \end{array}\right\}
=
\left\{\begin{array}{c} \{F\}_1 \\ -\{F\}_2 \end{array}\right\}
\qquad (2.5.4)
$$

As explained earlier, these equations can be rearranged as

$$
\left\{\begin{array}{c} \{u\}_2 \\ \{F\}_2 \\ 1 \end{array}\right\}
=
\left[\begin{array}{ccc}
-[A2]^{-1}[A1] & [A2]^{-1} & -[A2]^{-1}\{X1\} \\
-([A3]-[A4][A2]^{-1}[A1]) & -[A4][A2]^{-1} & ([X2]-[A4][A2]^{-1}\{X1\}) \\
\lfloor 0 \rfloor & \lfloor 0 \rfloor & 1
\end{array}\right.
$$

or

$$
\left\{\begin{array}{c} \{u\}_2 \\ \{F\}_2 \\ 1 \end{array}\right\}
= [TFM]_1
\left\{\begin{array}{c} \{u\}_1 \\ \{F\}_1 \\ 1 \end{array}\right\}
\qquad (2.5.5)
$$

Equation (2.5.5) gives the relation between the displacements and forces on the right side of the strip and the displacement and forces on the left hand side of the strip.

Case(b) : Domain with eight noded elements

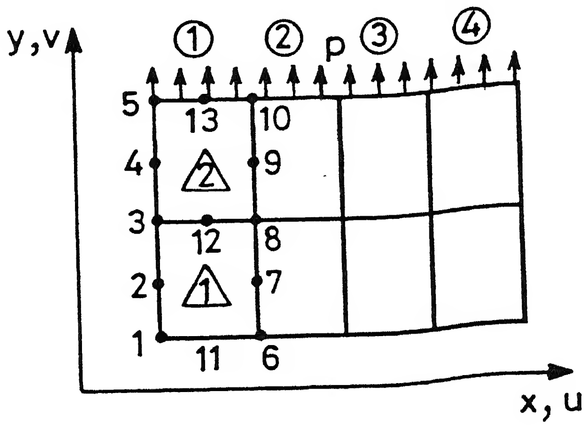


Fig. 2.5.2 : Rectangular lamina with pressure p applied on its upper edge

The finite element equations for the first strip are

$$
\begin{bmatrix}
k_{11} & k_{12} \cdots\cdots & k_{1,26} \\
k_{21} & k_{22} \cdots\cdots & k_{2,26} \\
k_{31} & k_{32} \cdots\cdots & k_{3,26} \\
k_{41} & k_{42} \cdots\cdots & k_{4,26} \\
\vdots & & \vdots \\
k_{26,1} & k_{26,2} & k_{26,26}
\end{bmatrix}
\begin{Bmatrix}
u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \\ u_7 \\ v_7 \\ u_8 \\ v_8 \\ u_9 \\ v_9 \\ u_{10} \\ v_{10} \\ u_{11} \\ v_{11} \\ u_{12} \\ v_{12} \\ u_{13} \\ v_{13}
\end{Bmatrix}
=
\begin{Bmatrix}
0 \\ 0 \\ \cdot \\ \cdot \\ \vdots \\ \cdot \\ 0
\end{Bmatrix}
+
\begin{Bmatrix}
Fx_{16} + Fx_{31} \\
Fy_{16} + Fy_{31} \\
Fx_{31} \\
Fy_{31} \\
Fx_{53} + Fx_{31} \\
Fy_{53} + Fy_{31} \\
Fx_{53} \\
Fy_{53} \\
Fx_{10,5} + Fx_{53} \\
Fy_{10,5} + Fy_{53} \\
Fx_{16} + Fy_{68} \\
Fy_{16} + Fy_{68} \\
Fx_{68} \\
Fy_{68} \\
Fx_{68} + Fx_{8,10} \\
Fy_{68} + Fy_{8,10} \\
Fx_{8,10} \\
Fy_{8,10} \\
Fx_{8,10} + Fx_{10,5} \\
Fy_{8,10} + Fy_{10,5} \\
Fx_{16} \\
Fy_{16} \\
0 \\
0 \\
Fx_{10,5} \\
Fx_{10,5}
\end{Bmatrix}
\qquad (2.5.6)
$$

Noting the boundary conditions on the bottom edge 16 ($Fx_{16} = Fy_{16} = 0$)and on the top edge 5,10 ($Fx_{5,10} = 0$; $Fy_{5,10}$ by Py_5, Py_{13} and Py_{10}), equations (2.5.6) become

$$
\left[\begin{array}{ccccc}
k_{11} & k_{12} & \cdots\cdots & k_{1,26} & 0 \\
k_{21} & k_{22} & \cdots\cdots & k_{2,26} & 0 \\
k_{31} & k_{32} & \cdots\cdots & k_{3,26} & 0 \\
k_{41} & k_{42} & \cdots\cdots & k_{4,26} & 0 \\
k_{51} & k_{52} & \cdots\cdots & k_{5,26} & 0 \\
k_{61} & k_{62} & \cdots\cdots & k_{6,26} & 0 \\
k_{71} & k_{72} & \cdots\cdots & k_{7,26} & 0 \\
k_{81} & k_{82} & \cdots\cdots & k_{8,26} & 0 \\
k_{91} & k_{92} & \cdots\cdots & k_{9,26} & 0 \\
k_{10,1} & k_{10,2} & \cdots\cdots & k_{10,26} & -Py_{5} \\
\hline
k_{11,1} & k_{11,2} & \cdots\cdots & k_{11,26} & 0 \\
k_{12,1} & k_{12,2} & \cdots\cdots & k_{12,26} & 0 \\
k_{13,1} & k_{13,2} & \cdots\cdots & k_{13,26} & 0 \\
k_{14,1} & k_{14,2} & \cdots\cdots & k_{14,26} & 0 \\
k_{15,1} & k_{15,2} & \cdots\cdots & k_{15,26} & 0 \\
k_{16,1} & k_{16,2} & \cdots\cdots & k_{16,26} & 0 \\
k_{17,1} & k_{17,2} & \cdots\cdots & k_{17,26} & 0 \\
k_{18,1} & k_{18,2} & \cdots\cdots & k_{18,26} & 0 \\
k_{19.1} & k_{19,2} & \cdots\cdots & k_{19,26} & 0 \\
k_{20,1} & k_{20,2} & \cdots\cdots & k_{20,26} & -Py_{10} \\
\hline
k_{21,1} & k_{21,2} & \cdots\cdots & k_{21,26} & 0 \\
k_{22,1} & k_{22,2} & \cdots\cdots & k_{22,26} & 0 \\
k_{23,1} & k_{23,2} & \cdots\cdots & k_{23,26} & 0 \\
k_{24,1} & k_{24,2} & \cdots\cdots & k_{24,26} & 0 \\
k_{25,1} & k_{25,2} & \cdots\cdots & k_{25,26} & 0 \\
k_{26,1} & k_{26,2} & \cdots\cdots & k_{26,26} & -Py_{13}
\end{array}\right]
\left\{\begin{array}{c}
u_{1} \\ v_{1} \\ u_{2} \\ v_{2} \\ u_{3} \\ v_{3} \\ u_{4} \\ v_{4} \\ u_{5} \\ v_{5} \\
\hline
u_{6} \\ v_{6} \\ u_{7} \\ v_{7} \\ u_{8} \\ v_{8} \\ u_{9} \\ v_{9} \\ u_{10} \\ v_{10} \\
\hline
u_{11} \\ v_{11} \\ u_{12} \\ v_{12} \\ u_{13} \\ v_{13} \\ 1
\end{array}\right\}
=
\left\{\begin{array}{c}
Fx_{31} \\ Fy_{31} \\ Fx_{31} \\ Fy_{31} \\ Fx_{53} + Fx_{31} \\ Fy_{53} + Fy_{31} \\ Fx_{53} \\ Fy_{53} \\ Fx_{53} \\ Fy_{53} \\
\hline
-Fx_{86} \\ -Fy_{86} \\ -Fx_{86} \\ -Fy_{86} \\ -Fy_{86} - Fx_{10,8} \\ -Fy_{86} - Fy_{10,8} \\ -Fx_{10,8} \\ -Fy_{10,8} \\ -Fx_{10,8} \\ -Fy_{10,8} \\
\hline
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{array}\right\}
\qquad (2.5.7)
$$

Since displacement at the intermediate nodes 11, 12, 13 are not required in the TFEM equations of the strip, these can be condensed out [19]. Thus partitioning the equations (2.5.7), and rewriting them,

$$\begin{bmatrix} \underset{10\times10}{[A1]} & \underset{10\times10}{[A2]} & \underset{10\times6}{[A3]} & \underset{10\times1}{\{X1]} \\ \\ \underset{10\times10}{[A4]} & \underset{10\times10}{[A5]} & \underset{10\times6}{[A6]} & \underset{10\times1}{\{X2\}} \\ \\ \underset{6\times10}{[A7]} & \underset{6\times10}{[A8]} & \underset{6\times6}{[A9]} & \underset{6\times1}{\{X3\}} \end{bmatrix} \left\{ \begin{matrix} \underset{10\times1}{\{u\}_1} \\ \\ \underset{10\times1}{\{u\}_2} \\ \\ \underset{6\times1}{\{u\}_3} \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} \underset{10\times1}{\{F\}_1} \\ \\ \underset{10\times1}{-\{F\}_2} \\ \\ \underset{6\times1}{\{0\}} \end{matrix} \right\} \tag{2.5.8}$$

Third equation of equations (2.5.8) gives $\{u\}_3$ in terms of $\{u\}_1$, and $\{u\}_2$. These can be substituted in first two equations of (2.5.8), one gets

$$\begin{bmatrix} ([A1]-[A3][A9]^{-1}[A7]) & ([A2]-[A3][A9]^{-1}[A8]) & (\{X1\}-[A3][A9]^{-1}\{X3\}) \\ -([A4]-[A6][A9]^{-1}[A7]) & -([A5]-[A6][A9]^{-1}[A8]) & -(\{X2\}-[A6][A9]^{-1}\{X3\}) \end{bmatrix}$$

$$\left\{ \begin{matrix} \{u\}_1 \\ \{u\}_2 \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} \{F\}_1 \\ \{F\}_2 \end{matrix} \right\} \tag{2.5.9}$$

or

$$\begin{bmatrix} [B1] & [B2] & \{B3\} \\ [B4] & [B5] & \{B6\} \end{bmatrix} \left\{ \begin{matrix} \{u\}_1 \\ \{u\}_2 \\ 1 \end{matrix} \right\} = \left\{ \begin{matrix} \{F\}_1 \\ \{F\}_2 \end{matrix} \right\} \tag{2.5.10}$$

These equations are similar to equations (2.5.9) and are rearranged in the same way as,

$$\begin{Bmatrix} \{u\}_2 \\ \{F\}_2 \\ 1 \end{Bmatrix} = \begin{bmatrix} -[B2]^{-1}[B1] & [B2]^{-1} & -[B2]^{-1}\{B3\} \\ ([B4]-[B5][B2]^{-1}[B1]) & [B5][B2]^{-1} & ([B6]-[B5][B2]^{-1}\{B3 \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & 1 \end{bmatrix} \begin{Bmatrix} \{u\}_1 \\ \{F\}_1 \\ 1 \end{Bmatrix} \qquad (2.5.11)$$

or

$$\begin{Bmatrix} \{u\}_2 \\ \{F\}_2 \\ 1 \end{Bmatrix} = [TFM]_1 \begin{Bmatrix} \{u\}_1 \\ \{F\}_1 \\ 1 \end{Bmatrix} \qquad (2.5.12)$$

These are the transfer FEM equations for the strip 1.

2.6 TFEM for heat conduction problems

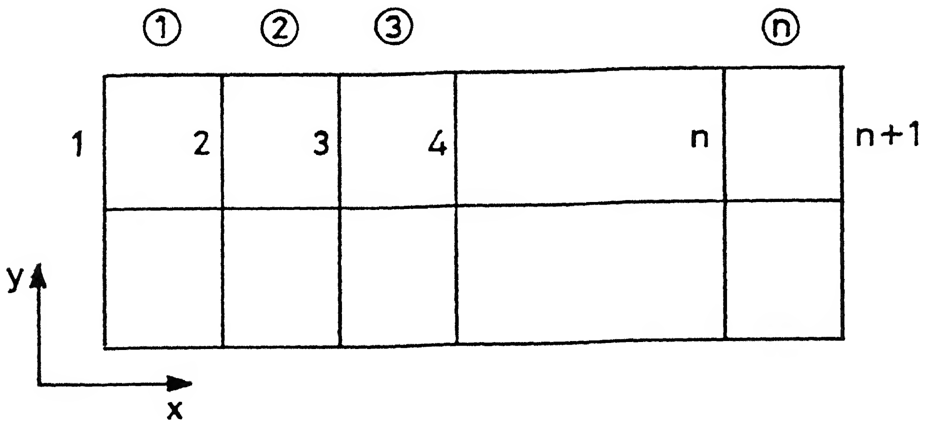


Fig. 2.6.1: Rectangular lamina divided into n-strips

In articles 2.3 and 2.4 transfer finite element equations were developed for certain heat conduction problems. For a typical strip, one get

$$\begin{Bmatrix} \{T\}_2 \\ \{q\}_2 \\ 1 \end{Bmatrix} = [TFM]_1 \begin{Bmatrix} \{T\}_1 \\ \{q\}_1 \\ 1 \end{Bmatrix} \qquad (2.6.1)$$

Referring to Fig. 2.6.1, for a domain with n strips, on repeated multiplications one reaches the last strip as,

$$\begin{Bmatrix} \{T\}_{n+1} \\ \{q\}_{n+1} \\ 1 \end{Bmatrix} = [TFM]_n \; [TFM]_{n-1} \cdots\cdots [TFM]_1 \begin{Bmatrix} \{T\}_1 \\ \{q\}_1 \\ 1 \end{Bmatrix}$$

or

$$\begin{Bmatrix} \{T\}_{n+1} \\ \{q\}_{n+1} \\ 1 \end{Bmatrix} = [TFM]_{final} \begin{Bmatrix} \{T\}_1 \\ \{q\}_1 \\ 1 \end{Bmatrix} = [z] \begin{Bmatrix} \{T\}_1 \\ \{q\}_1 \\ 1 \end{Bmatrix} \qquad (2.6.2)$$

Here $[z]$ is the final transfer matrix. Now the boundary conditions on the left side of the first strip and on the right side of the last strip are applied. Thus one can find the temperatures and heat fluxes. This is explained here for one set of boundary conditions, and extension to other sets of boundary conditions will be obvious. Here case (d) of article 2.3.3 is taken i.e. fluid on the upper edge and prescribed temperatures on the lower edge. Its TFEM equations will be similar to those of equation (2.3.26). Referring to Fig. 2.6.2 let the boundary conditions on the left hand edge the prescribed temperatures and on the right hand edge be insulation.

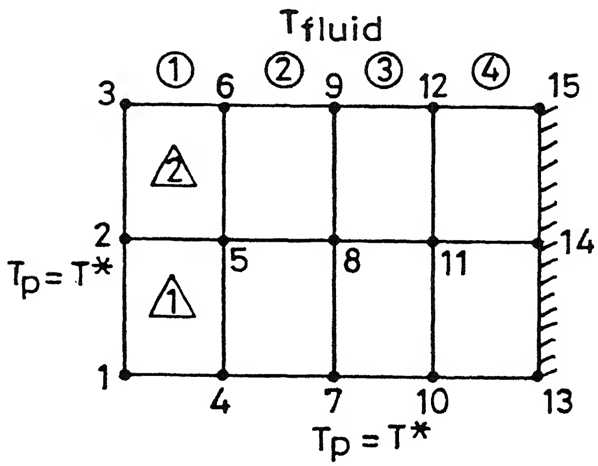


Fig. 2.6.2 : Rectangular lamina with boundary conditions prescribed on the first and the last edge.

Considering four strips in the domain the final TFEM equations (2.6.2) can be written as

$$\left\{\begin{matrix} T_{14} \\ T_{15} \\ ql|_{15,13} \\ ql|_{15,14} \\ 1 \end{matrix}\right\} = \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} \\ z_{41} & z_{42} & z_{43} & z_{44} & z_{45} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \left\{\begin{matrix} T_2 \\ T_3 \\ ql|_{31} \\ ql|_{32} \\ 1 \end{matrix}\right\} \tag{2.6.3}$$

Boundary conditions are $T_2 = T_2^*$, $T_3 = T_3^*$ and $ql|_{15,13} = 0$, $ql|_{15,14} = 0$. With these third and fourth equations give

$$\begin{Bmatrix} ql|_{31} \\ ql|_{32} \end{Bmatrix} = \begin{bmatrix} z_{33} & z_{34} \\ z_{43} & z_{44} \end{bmatrix}^{-1} \begin{Bmatrix} -z_{31}T_2^{*}-T_{32}T_3^{*}-z_{35} \\ -z_{41}T_2^{*}-z_{42}T_3^{*}-z_{45} \end{Bmatrix} \tag{2.6.4}$$

After having found the flux values $ql|_{31}$ and $ql|_{32}$, state vector on the left side of the first strip is completely known. Thus using the TFEM equations of each strip, starting with the first, one can find temperatures and fluxes everywhere.

2.7 TFEM for plane stress problems

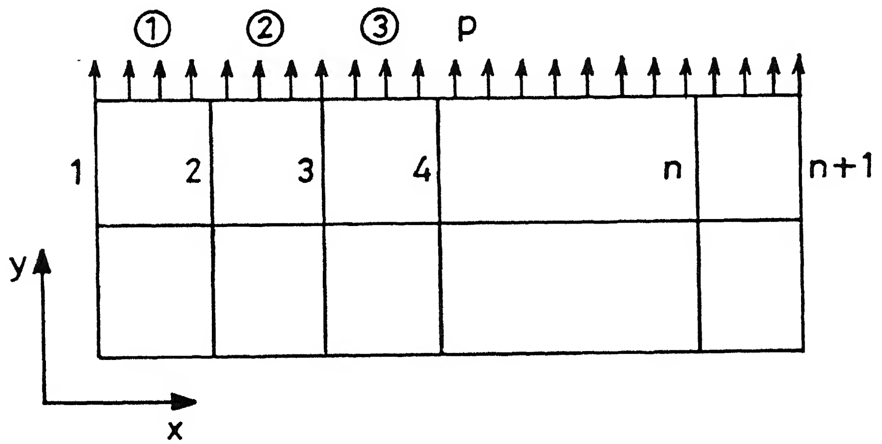


Fig. 2.7.1 : Rectangular lamina divided into n-strips and subjected to load p

In article 2.5 TFEM equations were developed for plane stress problems, see equation (2.5.5).

Referring to Fig. 2.7.1, for a domain with n-strips, on repeated multiplication one reaches the last strip as

$$\begin{Bmatrix} \{u\}_{n+1} \\ \{F\}_{n+1} \\ 1 \end{Bmatrix} = [TFM]_n \; [TFM]_{n-1} \quad [TFM]_1 \begin{Bmatrix} \{u\}_1 \\ \{F\}_1 \\ 1 \end{Bmatrix} \tag{2.7.1}$$

or

$$\left\{\begin{array}{c}\{u\}_{n+1}\\ \{F\}_{n+1}\\ 1\end{array}\right\} = [TFM]_{final} \left\{\begin{array}{c}\{u\}_1\\ \{F\}_1\\ 1\end{array}\right\} = [z]\left\{\begin{array}{c}\{u\}_1\\ \{F\}_1\\ 1\end{array}\right\} \quad (2.7.2)$$

Now the boundary conditions on the left side of the first strip and on the right side of the last strip are applied. Thus one can find the forces and the displacements. This is explained here for one set of boundary conditions, and extension to other sets of boundary conditions will be obvious.

Here case (a) of article 2.5 is taken. Let the first and the last edges be fixed in both directions, Fig. 2.7.2.

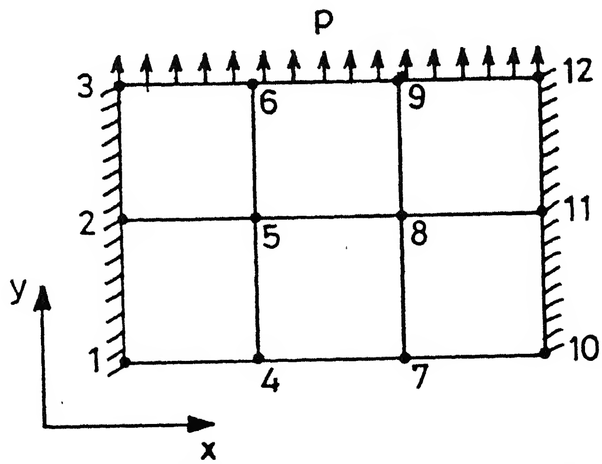


Fig. 2.7.2 : Rectangular plane with the first and the last edge fixed.

Taking three strips in the domain, the final transfer matrix equation (2.7.2) can be written as

$$\begin{Bmatrix} u_{10} \\ v_{10} \\ u_{11} \\ v_{11} \\ u_{12} \\ v_{12} \\ Fx_{11,10} \\ Fy_{11,10} \\ Fx_{12,10} \\ Fy_{12,10} \\ Fx_{12,11} \\ Fy_{12,11} \\ 1 \end{Bmatrix} = \begin{bmatrix} z_{11} & z_{12} & \cdots\cdots & z_{1,13} \\ z_{21} & z_{22} & \cdots\cdots & z_{2,13} \\ z_{31} & z_{32} & \cdots\cdots & z_{3,13} \\ z_{41} & z_{42} & \cdots\cdots & z_{4,13} \\ z_{51} & z_{52} & \cdots\cdots & z_{5,13} \\ z_{61} & z_{62} & \cdots\cdots & z_{6,13} \\ z_{71} & z_{72} & \cdots\cdots & z_{7,13} \\ z_{81} & z_{82} & \cdots\cdots & z_{8,13} \\ z_{91} & z_{92} & \cdots\cdots & z_{9,13} \\ z_{10,1} & z_{10,2} & \cdots\cdots & z_{10,13} \\ z_{11,1} & z_{11,2} & \cdots\cdots & z_{11,13} \\ z_{12,1} & z_{12,2} & \cdots\cdots & z_{12,13} \\ 0 & 0 & \cdots\cdots & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ Fx_{21} \\ Fy_{21} \\ Fx_{31} \\ Fy_{31} \\ Fx_{32} \\ Fy_{32} \\ 1 \end{Bmatrix} \tag{2.7.3}$$

Boundary conditions are $u_1 = v_1 = u_2 = v_2 = u_3 = v_3 = 0$ and $u_{10} = v_{10} = u_{11} = v_{11} = u_{12} = v_{12} = 0$. With these first six equations give

$$\begin{Bmatrix} -z_{1,13} \\ -z_{2,13} \\ -z_{3,13} \\ -z_{4,13} \\ -z_{5,13} \\ -z_{6,13} \end{Bmatrix} = \begin{bmatrix} z_{17} & z_{18} & z_{19} & z_{1,10} & z_{1,11} & z_{1,12} \\ z_{27} & z_{28} & z_{29} & z_{2,10} & z_{2,11} & z_{2,12} \\ z_{37} & z_{38} & z_{39} & z_{3,10} & z_{3,11} & z_{3,12} \\ z_{47} & z_{48} & z_{49} & z_{4,10} & z_{4,11} & z_{4,12} \\ z_{57} & z_{58} & z_{59} & z_{5,10} & z_{5,11} & z_{5,12} \\ z_{67} & z_{68} & z_{69} & z_{6,10} & z_{6,11} & z_{6,12} \end{bmatrix} \begin{Bmatrix} Fx_{21} \\ Fy_{21} \\ Fx_{31} \\ Fy_{31} \\ Fx_{32} \\ Fy_{32} \end{Bmatrix} \tag{2.7.4}$$

i.e. force vector on left side the first strip becomes known. Thus state vector on the left side of the first strip is completely known. Thus using the TFEM equations of each strip, starting with the first, one can find displacements and forces everywhere.

CHAPTER-3

THERMAL AND STRESS ANALYSIS OF AXI-SYMMETRIC PROBLEMS

Axi-symmetric bodies subjected to axi-symmetric loading often have complex geometrical features. For example a vertical pressure vessel with a spherical or ,hemi-spherical or ,tori-spherical or an elliptical head subjected to axi-symmetric thermal and mechanical loadings.At times such vessels have openings (inlet and outlet) in the centre of the head and thicknesses of the head and shell are different.

In this chapter TFEM equations will be developed for steady and transient heat conduction and stress analysis problems for axisymmetric bodies. Such problems are mathematically two dimensional problems.

3.1 Governing differential equation

The governing differential equation for heat conduction for axisymmetric problems are [23]

$$\frac{1}{r}\frac{\partial}{\partial r}\left(k_r\, r\,\frac{\partial T}{\partial r}\right) + \frac{\partial}{\partial z}\left(k_z\,\frac{\partial T}{\partial z}\right) = \rho\, c\,\frac{\partial T}{\partial t} \qquad (3.1.1)$$

with boundary conditions

$$T = T_o\,(r,z) \quad \text{at} \quad t = 0$$

and

$$k_r\, r\,\frac{\partial T}{\partial r} n_r + k_z r\,\frac{\partial T}{\partial z}\, n_z + q + h\,(T - T_f) = 0 \qquad (3.1.2)$$

3.2 FEM equations, heat conduction

Denoting the temperature difference $T - T_f$ by θ, the governing differential equation (3.1.1) can be rewritten as,

$$\frac{1}{r}\ \frac{\partial}{\partial r}\ (k_r\ r\ \frac{\partial\theta}{\partial r}) + \frac{\partial}{\partial z}(k_z\ \frac{\partial\theta}{\partial z}) = \rho\ c\ \frac{\partial\theta}{\partial t} \qquad (3.2.1)$$

and boundary conditions become

$$\theta = \theta_o\ (r,z) \quad \text{at} \quad t = 0$$

and

$$k_r\ r\ \frac{\partial\theta}{\partial r}\ n_r + k_z\ r\ \frac{\partial\theta}{\partial z}\ n_z + q + h\theta = 0 \qquad (3.2.2)$$

Assuming $\theta^{(e)}$ over the element

$$\theta^{(e)} = \lfloor N(r,z) \rfloor \{\theta(t)\}^{(ne)} \qquad (3.2.3)$$

and substituting in equation (3.2.1), one gets the residue as

$$R^{(e)} = \frac{1}{r}\ \frac{\partial}{\partial r}\ (k_r\ r\ \frac{\partial\theta}{\partial r}^{e)}) + \frac{\partial}{\partial z}(k_z\ \frac{\partial\theta}{\partial z}^{(e)}) - \rho c\frac{\partial\theta}{\partial t}^{(e)}$$

$$(3.2.4)$$

Minimizing this residue by Galerkin method, one gets

$$\int\int_{A^{(e)}} N_i \left[\frac{1}{r}\ \frac{\partial}{\partial r}\ (k_r.r.\ \frac{\partial\theta}{\partial r}^{e)}) + \frac{\partial}{\partial z}\ (k_z\ \frac{\partial\theta}{\partial z}^{(e)}) - \rho c\frac{\partial\theta}{\partial t}^{(e)}\right] r\ dr\ dz=0$$

Doing integration by parts, one gets,

$$\int\int_{A^{(e)}} \left[(k_r\ .r.\{N,r\}\ \frac{\partial\theta}{\partial r}^{(e)} + k_z.r.\ \{N,z\}\frac{\partial\theta}{\partial z}^{(e)}) - \rho\ cr\ \{N\}\ \frac{\partial\theta}{\partial t}^{(e)}\right] dr\ dz$$

$$= \oint -\ \{N\}\ (q + h\ \theta^{(e)})\ dB \qquad (3.2.5)$$

Substituting equation (3.2.3), these become

$$\int\int_{A^{(e)}} \Big[(k_r \, .r.\{N,r\}\lfloor N,r \rfloor + k_z \, r.\{N,z\} \lfloor N,z \rfloor) \, \{\theta\}^{(ne)}$$

$$+ \rho \, cr \, \{N\} \lfloor N \rfloor \, \{\dot{\theta}\}^{(ne)} \Big] dr \, dz = \oint - \{N\} \, (q + h \lfloor N \rfloor \, \theta^{(ne)}) \, dB \quad (3.2.6)$$

Equation (3.2.6) in short form is,

$$[C]^{(e)} \, \{\dot{\theta}\}^{(ne)} + [k_t]^{(e)} \, \{\theta\}^{(ne)} + [k_h]^{(e)} \, \{\theta\}^{(ne)} = \{q\}^{(ne)} \quad (3.2.7)$$

where

$[C]^{(e)} = \int\int \rho \, cr \, \{N\} \lfloor N \rfloor \, dr \, dz$ = Thermal capacitance matrix

$[k_t]^{(e)} = \int\int (k_r.r.\{N,r\} \lfloor N,r \rfloor + k_z.r.\{N,_z\}\lfloor N,_z \rfloor) \, dr \, dz$

= Thermal stiffness matrix

$[k_h]^{(e)} = \oint h\{N\} \lfloor N \rfloor \, dB$ = Elemental convection matrix where applicable

$\{q\}^{(ne)} = \oint - q.\{N\} \, dB$ = Elemental flux vector where applicable

For steady state heat conduction problems, equation (3.2.7) becomes,

$$[k_t] \, \{\theta\}^{(ne)} = \{q\}^{(ne)} \quad (3.2.8)$$

3.3 FEM equations, stress analysis

For stress analysis, variational approach is more convenient and is used here. The displacements in an eight noded element case are approximated over the eight noded element as,

$$\begin{Bmatrix} u \\ v \end{Bmatrix}^{(e)} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots\cdots & N_8 & 0 \\ 0 & N_1 & 0 & N_2 & \cdots\cdots & 0 & N_8 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_8 \\ v_8 \end{Bmatrix}$$

or

$$\{u\}^{(ne)} = [N] \quad \{u\}^{(ne)} \qquad (3.3.1)$$

The strain matrix is [22]

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial r \\ u/r \\ \partial v/\partial z \\ \frac{\partial u}{\partial z} + \frac{\partial v}{\partial r} \end{Bmatrix} = \left[\begin{array}{cc:cc:c:cc} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \cdots & \frac{\partial N_8}{\partial r} & 0 \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \cdots & \frac{N_8}{r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & \cdots & 0 & \frac{\partial N_8}{\partial z} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \cdots & \frac{\partial N_8}{\partial z} & \frac{\partial N_8}{\partial r} \end{array} \right] \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \\ u_8 \\ v_8 \end{Bmatrix}$$

$$(3.3.2)$$

and the stress matrix is written as

$$\{\sigma\} = [D] \; [\epsilon] \tag{3.3.3}$$

where $[D]$ is the elasticity matrix [21].

Using the above equations, the potential energy for axi-symmetric elements becomes

$$W^{(e)} = \frac{1}{2} \lfloor u \rfloor^{(ne)} \int\int_{A^{(e)}} [B]^T [D] [B] \, 2\pi r \, dr \, dz \, \{u\}^{(ne)}$$

$$- \lfloor u \rfloor^{(ne)} \int\int_{A^{(e)}} [N]^T \begin{Bmatrix} F_r \\ F_z \end{Bmatrix} 2\pi r \, drdz - \lfloor u \rfloor^{(ne)} \int [N]^T$$

$$\begin{Bmatrix} q_r \\ q_z \end{Bmatrix} 2\pi r \, dB \tag{3.3.4}$$

where F_r, F_z are body forces per unit volume and q_r, q_z are surface forces per area. Minimizing this potential energy one gets

$$\int\int_{A^{(e)}} [B]^T \; [D] \; [B] \; 2\pi r \; dr \; dz \; \{u\}^{(ne)} = \int\int_{A^{(e)}} [N]^T$$

$$\begin{Bmatrix} F_r \\ F_z \end{Bmatrix} 2\pi r \, dr \, dz + \int_{B^{(e)}} [N]^T \begin{Bmatrix} q_r \\ q_z \end{Bmatrix} 2\pi r dB \tag{3.3.5}$$

and this in short form is

$$[k]^{(e)} \{u\}^{(ne)} = \{P_B\}^{(ne)} + \{P_s\}^{(ne)} \quad (3.3.6)$$

where

$[k]^{(e)}$ is the elemental stiffness matrix

$\{P_B\}^{(ne)}$ is the elemental body force vector

$\{P_s\}^{(ne)}$ is the elemental surface (external or internal force vector)

3.4 TFEM equations for the steady state heat conduction problems

Equations (3.2.8) for this heat conduction are similar to equation (2.3.6). Thus TFEM will be similar to equations of article 2.5 for various types of boundary conditions.

3.5 TFEM equations for transient heat conduction problems

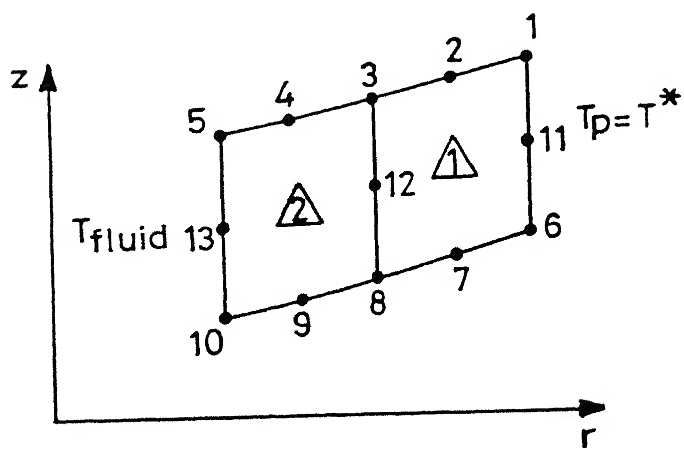


Fig.3.5 : Axi-symmetric strip with convection on the inside edge and temperature prescribed on the outer edge

Figure 3.5 shows a typical strip as used in the axi-symmetric problems, composed of two ring elements with fluid on the inside edge and prescribed temperature on the outer edge.

To deal with this transient heat conduction problem governed by equation (3.2.7), the use is made of the most successful scheme of 'Implicit Scheme', thus $\{\dot{\theta}\}^{(ne)}$ is represented as

$$\{\dot{\theta}\}^{(ne)} = \frac{\{\theta\}^{(ne)}_{p+1} - \{\theta\}^{(ne)}_{p}}{\Delta t} \qquad (3.5.1)$$

where p is the previous time

p+1 is the current time

Δt is the time step

Equations (3.2.7) become

$$(\frac{1}{\Delta t}[C] + [k_t] + (k_h]) \{\theta\}_{p+1} = \{q\}_{p+1} + \frac{1}{\Delta t} [C] \{\theta\}_p \qquad (3.5.2)$$

Using the Implicit scheme illustrated by equation (3.5.2), and noting convection on the edge 5-13-10, the FEM equations are given by

$$\left(\frac{1}{\Delta t}[C]+[k_t]+[k_h]\right)_{13\times 13}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\\ \theta_6\\ \theta_7\\ \theta_8\\ \theta_9\\ \theta_{10}\\ \theta_{11}\\ \theta_{12}\\ \theta_{13}\end{Bmatrix}_{p+1} = \begin{Bmatrix} ql|_{16}-ql|_{13}\\ -ql|_{13}\\ -ql|_{13}-ql|_{35}\\ -ql|_{35}\\ -ql|_{35}\\ ql|_{16}+ql|_{68}\\ ql|_{68}\\ ql|_{68}+ql|_{8,10}\\ ql|_{8,10}\\ ql|_{8,10}\\ ql|_{16}\\ 0\\ 0\end{Bmatrix}_{p+1} + \frac{1}{\Delta t}[C]\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\\ \theta_6\\ \theta_7\\ \theta_8\\ \theta_9\\ \theta_{10}\\ \theta_{11}\\ \theta_{12}\\ \theta_{13}\end{Bmatrix}_p$$

(3.5.3)

The state vector $\{\theta\}_p$ on the RHS in equation (3.5.3) is known from initial conditions. Denoting the matrix $(\frac{1}{\Delta t}[C]+[k_t]+[k_h])$ by $[r]$, and the vector $\frac{1}{\Delta t}[C]\ \{\theta\}_p$ by $\{x\}$, the finite element equations (3.5.3) after applying boundary conditions of the outer edge and rearranging become

$$
\begin{bmatrix}
r_{12} \cdots\cdots r_{1,13} & (r_{11}\theta_1^* + r_{1,11}\theta_{11}^* + r_{16}\theta_6^* - x_1) \\
r_{22} \cdots\cdots r_{2,13} & (r_{21}\theta_1^* + r_{2,11}\theta_{11}^* + r_{26}\theta_6^* - x_2) \\
\vdots \qquad \vdots & \\
r_{13,2} \qquad r_{13,13} & (r_{13,1}\theta_1^* + r_{13,11}\theta_{11}^* + r_{13,6}\theta_6^* - x_{13})
\end{bmatrix}_{13\times 11}
\left\{\begin{matrix} \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_7 \\ \theta_8 \\ \theta_9 \\ \theta_{10} \\ \theta_{12} \\ \theta_{13} \\ 1 \end{matrix}\right\}_{11\times 1}^{p+1}
$$

$$
= \left\{\begin{matrix}
ql|_{16} - ql|_{13} \\
-ql|_{13} \\
-ql|_{13} - ql|_{35} \\
-ql|_{35} \\
-ql|_{35} \\
ql|_{16} + ql|_{68} \\
ql|_{68} \\
ql|_{68} + ql|_{8,10} \\
ql|_{8,10} \\
ql|_{8,10} \\
ql|_{16} \\
0 \\
0
\end{matrix}\right\}_{13\times 1}^{p+1} \qquad (3.5.4)
$$

TFEM equations for this strip will be developed from second, third, fourth, fifth, seventh, eighth, ninth, tenth, twelfth and thirteenth equations. Any one of the dropped out equations can later be used to find the flux along edge 1-11-6.

Rewriting these equations,

$$
\begin{bmatrix}
r_{22} \cdots\cdots\cdots r_{2,13} & (r_{21}\theta_1^* + r_{2,11}\theta_{11}^* + r_{26}\theta_6^* - x_2) \\
r_{32} \cdots\cdots\cdots r_{3,13} & (r_{31}\theta_1^* + r_{3,11}\theta_{11}^* + r_{36}\theta_6^* - x_3) \\
r_{42} \cdots\cdots\cdots r_{4,13} & (r_{41}\theta_1^* + r_{4,11}\theta_{11}^* + r_{46}\theta_6^* - x_4) \\
r_{52} \cdots\cdots\cdots r_{5,13} & (r_{51}\theta_1^* + r_{5,11}\theta_{11}^* + r_{56}\theta_6^* - x_5) \\
\hdashline
r_{72} \cdots\cdots\cdots r_{7,13} & (r_{71}\theta_1^* + r_{7,11}\theta_{11}^* + r_{76}\theta_6^* - x_7) \\
r_{82} \cdots\cdots\cdots r_{8,13} & (r_{81}\theta_1^* + r_{8,11}\theta_{11}^* + r_{86}\theta_6^* - x_8) \\
r_{92} \cdots\cdots\cdots r_{9,13} & (r_{91}\theta_1^* + r_{9,11}\theta_{11}^* + r_{96}\theta_6^* - x_9) \\
r_{10,2} \cdots\cdots\cdots r_{10,13} & (r_{10,1}\theta_1^* + r_{10,11}\theta_{11}^* + r_{10,6}\theta_6^* - x_{10}) \\
\hdashline
r_{12,2} \cdots\cdots\cdots r_{12,13} & (r_{12,1}\theta_1^* + r_{12,11}\theta_{11}^* + r_{12,6}\theta_6^* - x_{12}) \\
r_{13,2} \cdots\cdots\cdots r_{13,13} & (r_{13,1}\theta_{11}^* + r_{13,11}\theta_{11}^* + r_{13,6}\theta_6^* - x_{13})
\end{bmatrix}_{10\times 11}
\left\{
\begin{bmatrix}
\theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \hdashline \theta_7 \\ \theta_8 \\ \theta_9 \\ \theta_{10} \\ \hdashline \theta_{12} \\ \theta_{13} \\ 1
\end{bmatrix}_{11\times 1}
\right\}^{p+1}
$$

$$
= \left\{
\begin{bmatrix}
-ql|_{13} \\ -ql|_{15} \\ -ql|_{35} \\ -ql|_{35} \\ \hdashline ql|_{68} \\ ql|_{6,10} \\ ql|_{8,10} \\ ql|_{8,10} \\ \hdashline 0 \\ 0
\end{bmatrix}_{10\times 1}
\right\}^{p+1}
\qquad (3.5.5)
$$

These equations are similar to eqns. (2.5.9) and TFEM can be developed in a similar manner.

$$\left\{ \begin{array}{c} \{\theta\}_2 \\ \{q\}_2 \\ 1 \end{array} \right\}_{p+1} = [TFM]_1 \left\{ \begin{array}{c} \{\theta\}_1 \\ \{q\}_1 \\ 1 \end{array} \right\}_{p+1} \qquad (3.5.6)$$

Thus $\{\theta\}$ and $\{q\}$ can be determined at all nodes at current time. Thus procedure is repeated for the next time step and so on.

3.6 TFEM equations for stress analysis

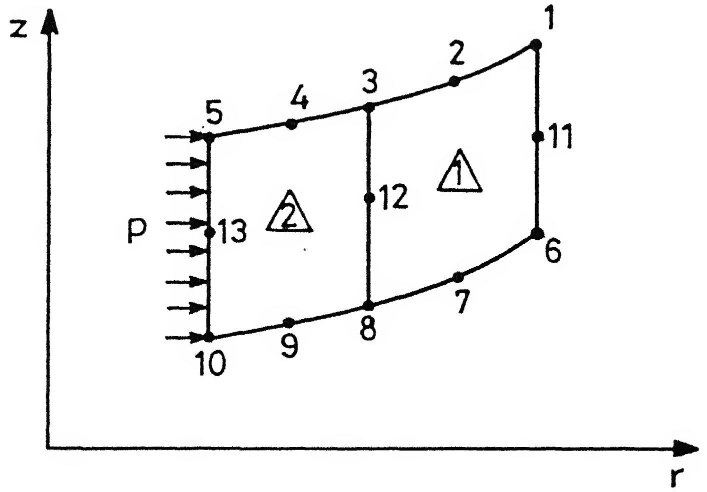


Fig. 3.6 : Axisymmetric strip with loading p on the inside edge

Figure 3.6 shows an axisymmetric strip subjected to a load p on its inside edge. The formulation for this case is quite similar to the case (b) of article 2.6. Rewriting the FEM equations for this strip after applying the boundary conditions along the edge 1-11-6 ($Fr_{16} = Fz_{16} = 0$) and the edge 5-13-10

$(Fz_{5,10} = 0;\ Fr_5 = Pr_5,\ Fr_{13} = Pr_{13}$ and $Fr_{10} = Pr_{10})$.

$$
\underset{26\times27}{\begin{bmatrix}
k_{11} & k_{12} & \cdots\cdots & k_{1,26} & 0 \\
k_{21} & k_{22} & \cdots\cdots & k_{2,26} & 0 \\
k_{31} & k_{32} & \cdots\cdots & k_{3,26} & 0 \\
k_{41} & k_{42} & \cdots\cdots & k_{4,26} & 0 \\
k_{51} & k_{52} & \cdots\cdots & k_{5,26} & 0 \\
k_{61} & k_{62} & \cdots\cdots & k_{6,26} & 0 \\
k_{71} & k_{72} & \cdots\cdots & k_{7,26} & 0 \\
k_{81} & k_{82} & \cdots\cdots & k_{8,26} & \\
k_{91} & k_{92} & \cdots\cdots & k_{9,26} & -Pr_5 \\
k_{10,1} & k_{10,2} & \cdots\cdots & k_{10,26} & 0 \\
\hdashline
k_{11,1} & k_{11,2} & \cdots\cdots & k_{11,26} & 0 \\
k_{12,1} & k_{12,2} & \cdots\cdots & k_{12,26} & 0 \\
k_{13,1} & k_{13,2} & \cdots\cdots & k_{13,26} & 0 \\
k_{14,1} & k_{14,2} & \cdots\cdots & k_{14,26} & 0 \\
k_{15,1} & k_{15,2} & \cdots\cdots & k_{15,26} & 0 \\
k_{16,1} & k_{16,2} & \cdots\cdots & k_{16,26} & 0 \\
k_{17,1} & k_{17,2} & \cdots\cdots & k_{17,26} & 0 \\
k_{18,1} & k_{18,2} & \cdots\cdots & k_{18,26} & 0 \\
k_{19,1} & k_{19,2} & \cdots\cdots & k_{19,26} & -Pr_{10} \\
k_{20,1} & k_{20,2} & \cdots\cdots & k_{20,26} & 0 \\
\hdashline
k_{21,1} & k_{21,2} & \cdots\cdots & k_{21,26} & 0 \\
k_{22,1} & k_{22,2} & \cdots\cdots & k_{22,26} & 0 \\
k_{23,1} & k_{23,2} & \cdots\cdots & k_{23,26} & 0 \\
k_{24,1} & k_{24,2} & \cdots\cdots & k_{24,26} & 0 \\
k_{25,1} & k_{25,2} & \cdots\cdots & k_{25,26} & -Pr_{13} \\
k_{26,1} & k_{26,2} & \cdots\cdots & k_{26,26} & 0
\end{bmatrix}}
\underset{27\times1}{\begin{Bmatrix}
u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\
\hdashline
u_6 \\ v_6 \\ u_7 \\ v_7 \\ u_8 \\ v_8 \\ u_9 \\ v_9 \\ u_{10} \\ v_{10} \\
\hdashline
u_{11} \\ v_{11} \\ u_{12} \\ v_{12} \\ u_{13} \\ v_{13} \\ 1
\end{Bmatrix}}
=
\underset{26\times1}{\begin{Bmatrix}
Fr_{13} \\ Fz_{13} \\ Fr_{13} \\ Fz_{13} \\ Fr_{15} \\ Fz_{15} \\ Fr_{35} \\ Fz_{35} \\ Fr_{35} \\ Fz_{35} \\
\hdashline
-Fr_{68} \\ -Fz_{68} \\ -Fr_{68} \\ -Fz_{68} \\ -Fr_{6,10} \\ -Fz_{6,10} \\ -Fr_{8,10} \\ -Fz_{8,10} \\ -Fr_{8,10} \\ -Fz_{8,10} \\
\hdashline
0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0
\end{Bmatrix}}
\tag{3.6.1}
$$

Since displacements at the intermediate nodes (11,12,13) are not required in the TFEM equations of the strip, these can be condensed out.

Partitioning equations (3.6.1) as indicated and rewriting them.

$$\begin{bmatrix} \underset{10\times 10}{[A1]} & \underset{10\times 10}{[A2]} & \underset{10\times 6}{[A3]} & \underset{10\times 1}{\{X1\}} \\ \underset{10\times 10}{[A4]} & \underset{10\times 10}{[A5]} & \underset{10\times 6}{[A6]} & \underset{10\times 1}{\{X2\}} \\ \underset{6\times 10}{[A7]} & \underset{6\times 10}{[A8]} & \underset{6\times 6}{[A9]} & \underset{6\times 1}{\{X3]} \end{bmatrix} \begin{Bmatrix} \underset{10\times 1}{\{u\}_1} \\ \underset{10\times 1}{\{u\}_2} \\ \underset{\substack{6\times 1 \\ 1}}{\{u\}_3} \end{Bmatrix} = \begin{Bmatrix} \underset{10\times 1}{\{F\}_1} \\ \underset{10\times 1}{-\{F\}_2} \\ \underset{6\times 1}{\{0\}} \end{Bmatrix} \tag{3.6.2}$$

After condensing $\{u\}_3$, the first two equations of equation (3.6.2) can be written as,

$$\begin{bmatrix} \underset{10\times 10}{[B1]} & \underset{10\times 10}{[B2]} & \underset{10\times 1}{[B3]} \\ \underset{10\times 10}{[B4]} & \underset{10\times 10}{[B5]} & \underset{10\times 1}{[B6]} \end{bmatrix} \begin{Bmatrix} \underset{10\times 1}{\{u\}_1} \\ \underset{\substack{10\times 1 \\ 1}}{\{u\}_2} \end{Bmatrix} = \begin{Bmatrix} \underset{10\times 1}{\{F\}_1} \\ \underset{10\times 1}{\{F\}_2} \end{Bmatrix} \tag{3.6.3}$$

One finally gets the transfer matrix as,

$$\begin{Bmatrix} \{u\}_2 \\ \{F\}_2 \\ 1 \end{Bmatrix} = \begin{bmatrix} -[B2]^{-1}\,[B1] & [B2]^{-1} & -[B2]^{-1}\{B3\} \\ ([B4]-[B5][B2]^{-1}[B1]) & [B5][B2]^{-1} & ([B6]-[B5][B2]^{-1}\{B \\ \lfloor 0 \rfloor & \lfloor 0 \rfloor & 1 \end{bmatrix} \begin{Bmatrix} \{u\}_1 \\ \{F\}_1 \\ 1 \end{Bmatrix} \qquad (3.6.4)$$

or

$$\begin{Bmatrix} \{u\}_2 \\ \{F\}_2 \\ 1 \end{Bmatrix} = [TFM]_1 \begin{Bmatrix} \{u\}_1 \\ \{F\}_1 \\ 1 \end{Bmatrix} \qquad (3.6.5)$$

3.7 TFEM equation for heat conduction for a general case

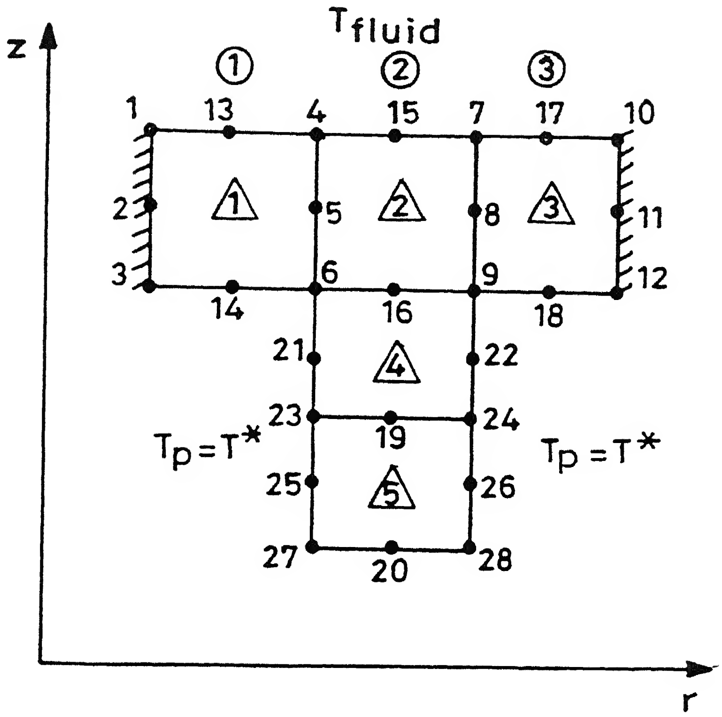


Fig. 3.7 : A general axisymmetric case

Fig. 3.7 shows a general axisymmetric case. Development of TFEM equations in such problems is different from the previous cases, owing to the presence of more elements in some strips than other strips. Here strip 2 is shown to have two additional elements in the z-direction. The additional nodes in such strips are condensed out.

TFEM equations for strip 1 after applying boundary conditions along edge 14 and edge 36 are

$$\left\{\begin{bmatrix} \theta_4 \\ \theta_5 \\ ql|_{46} \\ ql|_{46} \\ 1 \end{bmatrix}\right\} = [TFM]_1 \left\{\begin{bmatrix} \theta_1 \\ \theta_2 \\ ql|_{13} \\ ql|_{13} \\ 1 \end{bmatrix}\right\} \qquad (3.7.1)$$

Finite element equations of strip 2, after applying the boundary conditions of edges 47, 6-27, 27-28 and 28-9, are

$$\begin{bmatrix} t_{44} & t_{45} \cdots t_{4,19} & (\theta_6^* t_{46} + \theta_{21}^* t_{4,21} + \theta_{23}^* t_{4,23} + \cdots + \theta_9^* t_{4,9}) \\ t_{54} & t_{55} \cdots t_{5,19} & (\theta_6^* t_{56} + \theta_{21}^* t_{5,21} + \theta_{23}^* t_{5,23} + \cdots + \theta_9^* t_{5,9}) \\ t_{74} & t_{75} \cdots t_{7,19} & (\theta_6^* t_{76} + \theta_{21}^* t_{7,21} + \theta_{23}^* t_{7,23} + \cdots + \theta_9^* t_{7,9}) \\ t_{84} & t_{85} \cdots t_{8,19} & (\theta_6^* t_{86} + \theta_{21}^* t_{8,21} + \theta_{23}^* t_{8,23} + \cdots + \theta_9^* t_{8,9}) \\ t_{15,4} & t_{15,5} \cdots t_{15,19} & (\theta_6^* t_{15,6} + \theta_{21}^* t_{15,21} + \theta_{23}^* t_{15,23} + \cdots + \theta_9^* t_{15,9}) \\ t_{16,4} & t_{16,5} \cdots t_{16,19} & (\theta_6^* t_{16,6} + \theta_{21}^* t_{16,21} + \theta_{23}^* t_{16,23} + \cdots + \theta_9^* t_{16,9}) \\ t_{19,4} & t_{19,5} \cdots t_{19,19} & (\theta_6^* t_{19,6} + \theta_{21}^* t_{19,21} + \theta_{23}^* t_{19,23} + \cdots + \theta_9^* t_{19,9}) \end{bmatrix}_{7\times 8}$$

$$\begin{Bmatrix} \theta_4 \\ \theta_5 \\ \theta_7 \\ \theta_8 \\ \theta_{15} \\ \theta_{16} \\ \theta_{19} \\ 1 \end{Bmatrix}_{8\times 1} = \begin{Bmatrix} -ql|_{46} \\ -ql|_{46} \\ ql|_{79} \\ ql|_{79} \\ 0 \\ 0 \\ 0 \end{Bmatrix}_{7\times 1} \qquad (3.7.2)$$

Since strip 2 contains additional intermediate nodes, so

they will be condensed out to make TFEM equations compatible for marching. Thus condensing nodes 15,16 and 19, the TFEM equations for strip 2 are:

$$\begin{Bmatrix} \theta_7 \\ \theta_8 \\ ql|_{79} \\ ql|_{79} \\ 1 \end{Bmatrix} = [TFM]_2 \begin{Bmatrix} \theta_4 \\ \theta_5 \\ ql|_{46} \\ ql|_{46} \\ 1 \end{Bmatrix} \tag{3.7.3}$$

Similarly TFEM equations for strip 3 are given as,

$$\begin{Bmatrix} \theta_{10} \\ \theta_{11} \\ ql|_{10-12} \\ ql|_{10-12} \\ 1 \end{Bmatrix} = [TFM]_3 \begin{Bmatrix} \theta_7 \\ \theta_8 \\ ql|_{79} \\ ql|_{79} \\ 1 \end{Bmatrix} \tag{3.7.4}$$

Now transfer matrices of strips i.e. equations (3.7.1), (3.7.3) and (3.7.4) can be multiplied and results obtained as usual.

3.8 TFEM for axisymmetric problems

TFEM equations for a typical strip have been derived in articles 3.4, 3.5, 3.6 and 3.7 for axi-symmetric heat conduction and stress analysis problems. Now one can multiply the transfer matrices for various strips and get all the results after applying the boundary conditions on the first and last edge, as explained in Chapter 2.

CHAPTER - 4

RESULTS AND DISCUSSIONS

Computer programs in FORTRAN language have been made for plane lamina and axi-symmetric thermal and stress analysis problems incorporating different boudary conditions. A rectangular domain is considered in case of plane lamina problems and a pressure vessel with hemi-spherical heads is considered is case of axi-symmetric problems.

4.1 Thermal analysis of rectangular lamina with four noded rectangular elements

To validate the TFEM, first results were obtained for rectangular laminas with four noded rectangular elements. Each lamina was 5x10 m. Coefficient of conductivity was taken as 54 W/m.^{o}C. Each lamina was divided into ten strips and each strip in five elements, as shown in Figs.4.1.1, 4.1.2, 4.1.3 and 4.1.4. As said in Chapter 2, following boundary conditions are considered on top and bottom edges.

(a) Top and bottom edges insulated.

(b) Top edge with prescribed temperature (200^{o}) and bottom edge insulated.

(c) Top and bottom edges with prescribed temperatures i.e. 300^{o}C at top and 100^{o}C at bottom.

(d) Top edge exposed to fluid (convection) at 100^{o} and bottom edge with prescribed temperature (10^{o}C).

Following boundary conditions are considered on the first and

last edges for the above four cases.

(a) First edge with prescribed temperature 200°C and last edge with prescribed temperature of 10°C.

(b) First edge insulated and last edge with prescribed temperature of 100°C.

(c) First and last edge with prescribed temperature of 100°C.

(d) First and last edge insulated.

Figure 4.1.1 shows results for case (a). Temperatures along the vertical edges of strip are identical, as this problem is essentially one dimensional heat flow in x-direction. These results obtained by TFEM match exactly with the analytical results [20].

Results for case (b) are shown in Fig. 4.1.2. This is a two dimensional problem. Temperature decrease from left to right and also from top to bottom, as expected. Results for case (c) are shown in Fig. 4.1.3. This is also a two dimensional problem. Temperatures are symmetric about the middle vertical edge, and are maximum along this edge and decrease on both sides (left and right); as temperatures are same on the first and last edge. These temperatures decrease from top to bottom as expected.

Temperatures for case (d) are shown in Fig. 4.1.4. Heat transfer coefficient for this case is taken as 50 $W/m^2.{}^oC$. Temperatures along the horizontal edges of the strip are identical, as the problem is essentially one dimensional heat flow in y-direction. These results by TFEM match exactly with the analytical results [20].

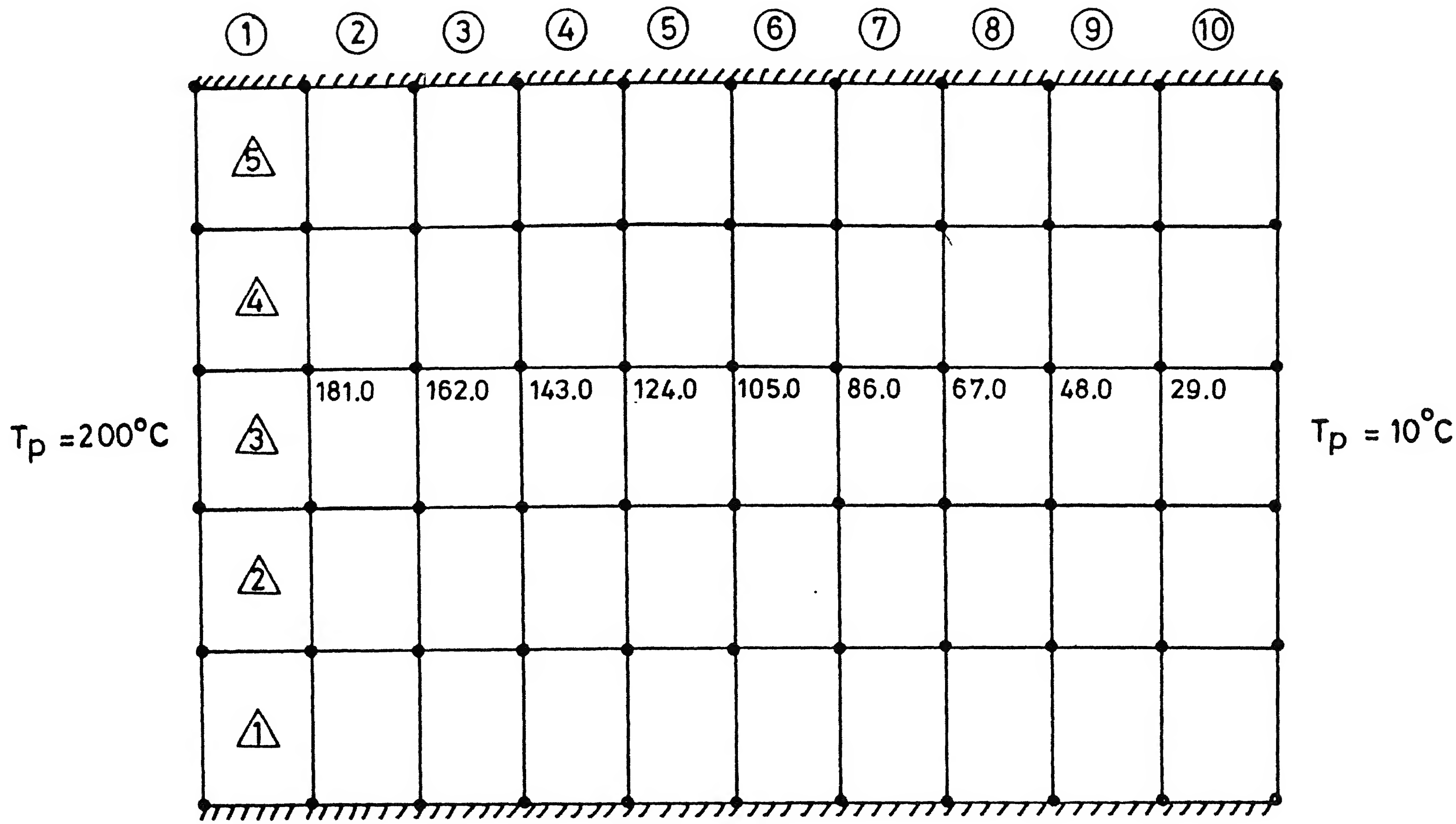


Fig. 4.1.1 Top and bottom edge insulated .

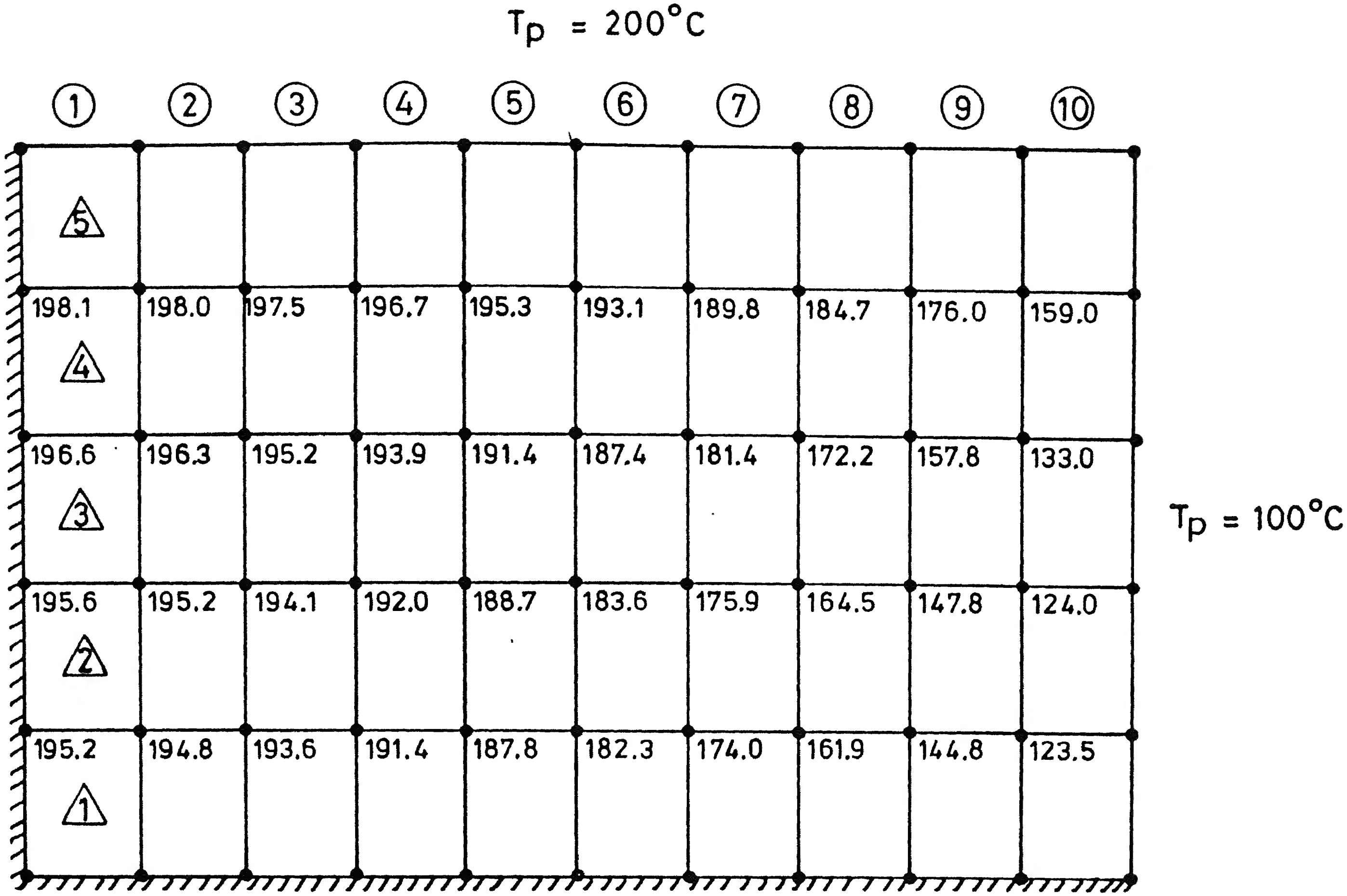


Fig. 4.1.2 Top edge with prescribed temperature and bottom edge insulated.

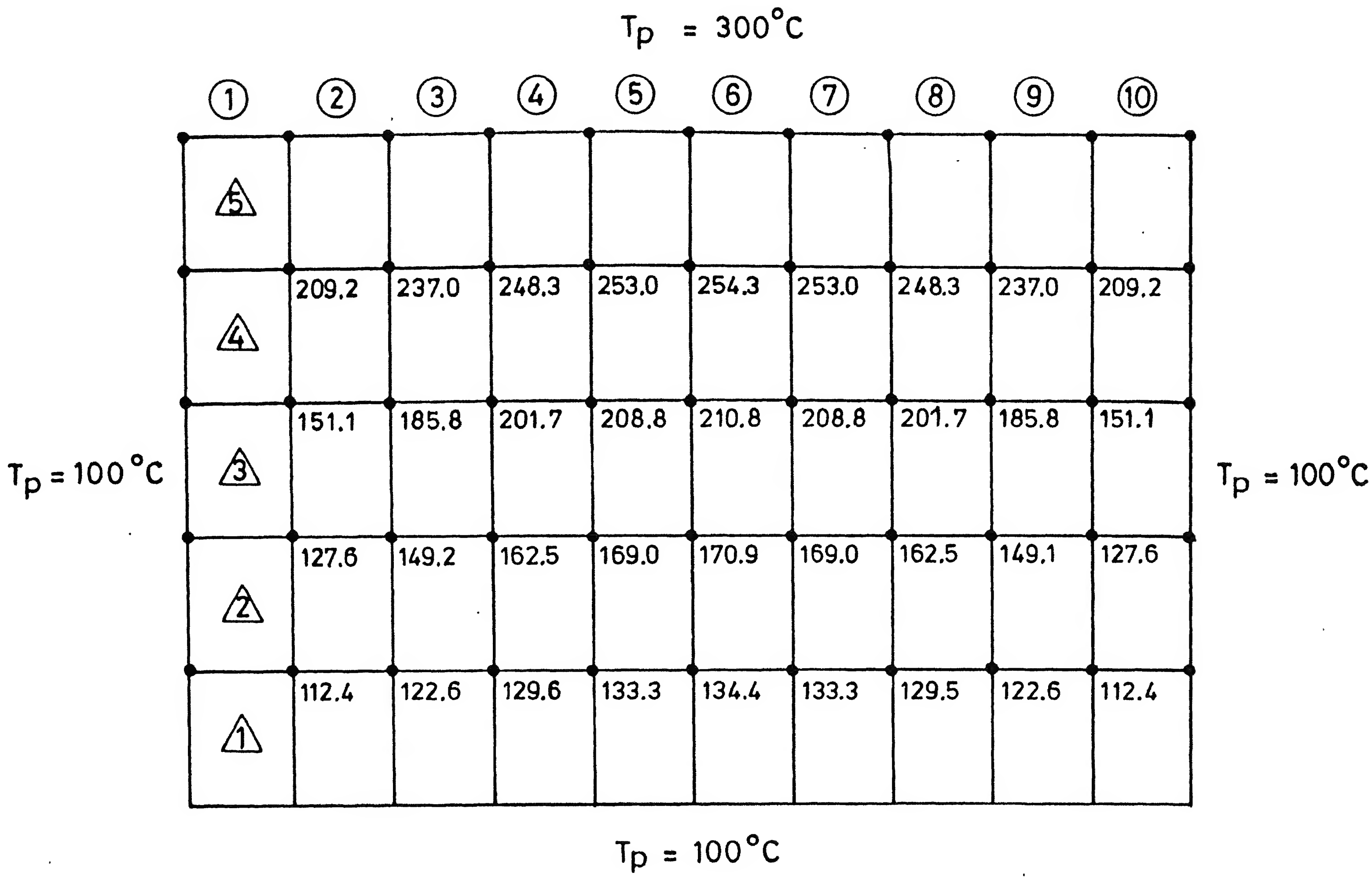


Fig. 4.1.3 Top and bottom edges with prescribed temperatures.

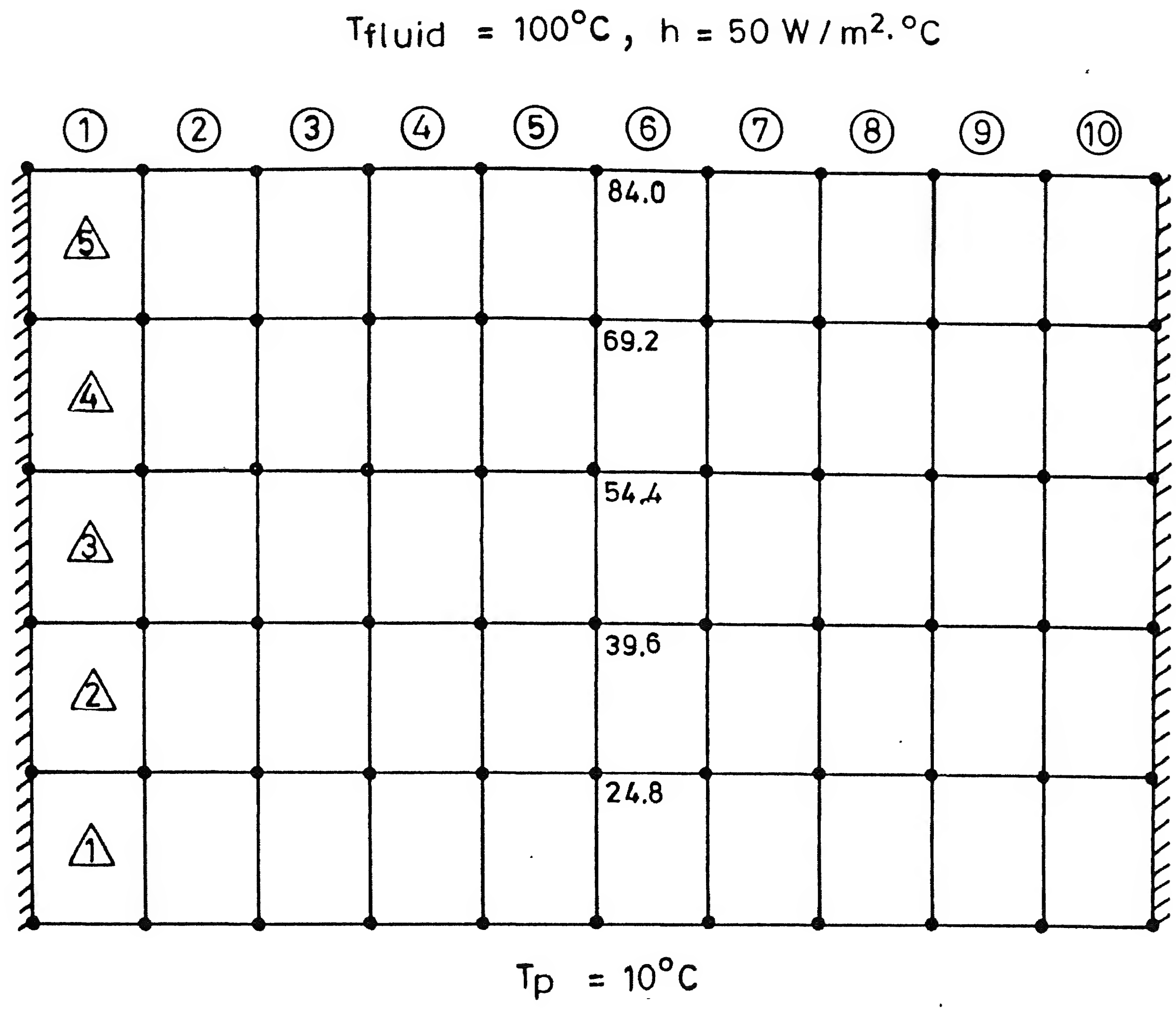


Fig. 4.1.4 Top edge exposed to fluid (air) and bottom edge with prescribed temperature.

4.2 Thermal steady and transient analysis of pressure vessel with eight noded elements

Vertical pressure vessel with hemispherical heads shown in Figure 4.2.1 is studied. Inside diameter is 2000 mm, thickness is 100 mm and outside length is 4200 mm.

First following two cases of steady state heat conduction are studied.

(i) Temperature prescribed at inner surface as 100°C and at outer surface as 10°C.

(ii) Inside surface exposed to fluid at 250°C and outside surface with a prescribed temperature of 30°C.

Coefficient of conductivity was taken as 54 W/m.°C and coefficient of convective heat transfer as 5000 $W/m^2.°C$. It was noted that because of symmetry there will be no heat flow across the lines r_1 and r_2, i.e. $q = 0$. These are the boundary conditions for the first and last edge.

Vessel was divided into 10 strips with two eight noded elements in each strip, as shown in Figure 4.2.1.

Temperatures obtained by TFEM for case (i) are shown in Fig. 4.2.1. These are symmetric with respect to the central longitudinal axis of the pressure vessel and also w.r.t line r_3 as expected. Temperatures at node points along the line r_1 in the head by TFEM are 75.832, 52.845, 30.911 and by analytical method applied to sphere alone are 75.853, 52.857, 30.930. Temperature at node points along line r_3 in the shell by TFEM are 76.650,

53.905, 31.679 and by analytical method applied to cylinder alone are 76.683, 53.938, 31.708. Also temperatures decrease along the radial line, as expected.

Temperatures obtained by TFEM for case (ii) are shown in Fig. 4.2.2. They follow the trend of case (i). Temperatures at inside surface of the pressure vessel are less than the fluid temperature as expected. Values at the nodes along the line r_1 by TFEM are 227.279, 174.335, 123.997, 75.874 and by analytical method for sphere alone are 226.641, 173.884, 123.640, 75.732. Similar results for line r_2 are 227.605, 176.340, 126.401, 77.600, 227.608, 176.412, 126.450, 77.664. Thus it is concluded that TFEM results are quite accurate.

Next case (ii) was studied again for a pressure vessel with skirt, as shown in Fig. 4.2.3. Here vessel is again divided into 10 strips with the eight noded elements in each strip except in the strip 8 (having skirt) which has five eight noded elements. Results are shown in Fig. 4.2.3. These are quite close to that of the vessel without skirt, Fig. 4.2.2, except in the strip 8 where results change quite a bit due to the effect of the skirt. Moreover the results are not symmetric with respect to the line r_3, as vessel is no more symmetric with respect to this line.

Next transient heat conduction was studied in the pressure vessel of Fig. 4.2.1. Vessel has fluid at 250^{o}C inside and its outside surface is maintained at 30^{o}C. Value of ρ is taken as 7833 kg/m^3 and specific heat C as 465 J/kg.^{o}C. Time steps considered were 5, 10, 20 and 30 seconds. Time steps of 5 and 10

seconds were leading to numerical instability after some time. Time steps 20 and 30 seconds were found satisfactory. Results are shown in Figure 4.2.4 at 20,60, 100, 300, 400 and 500 seconds using the time step of 20 seconds.They show the expected trend. At 500 seconds the temperatures are nearly the steady state values.

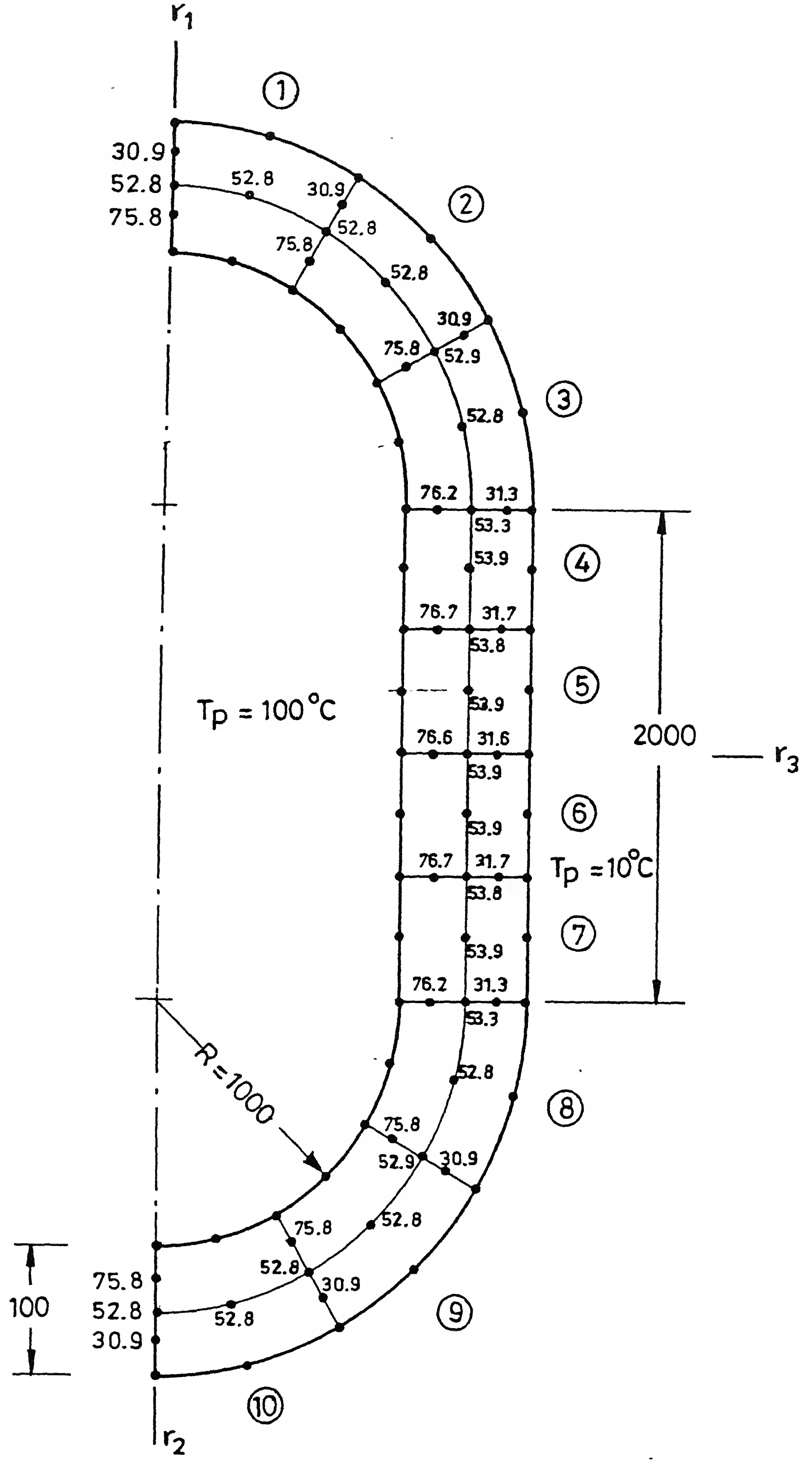


Fig. 4.2.1 Pressure vessel with prescribed temperatures at inside and out side surfaces.

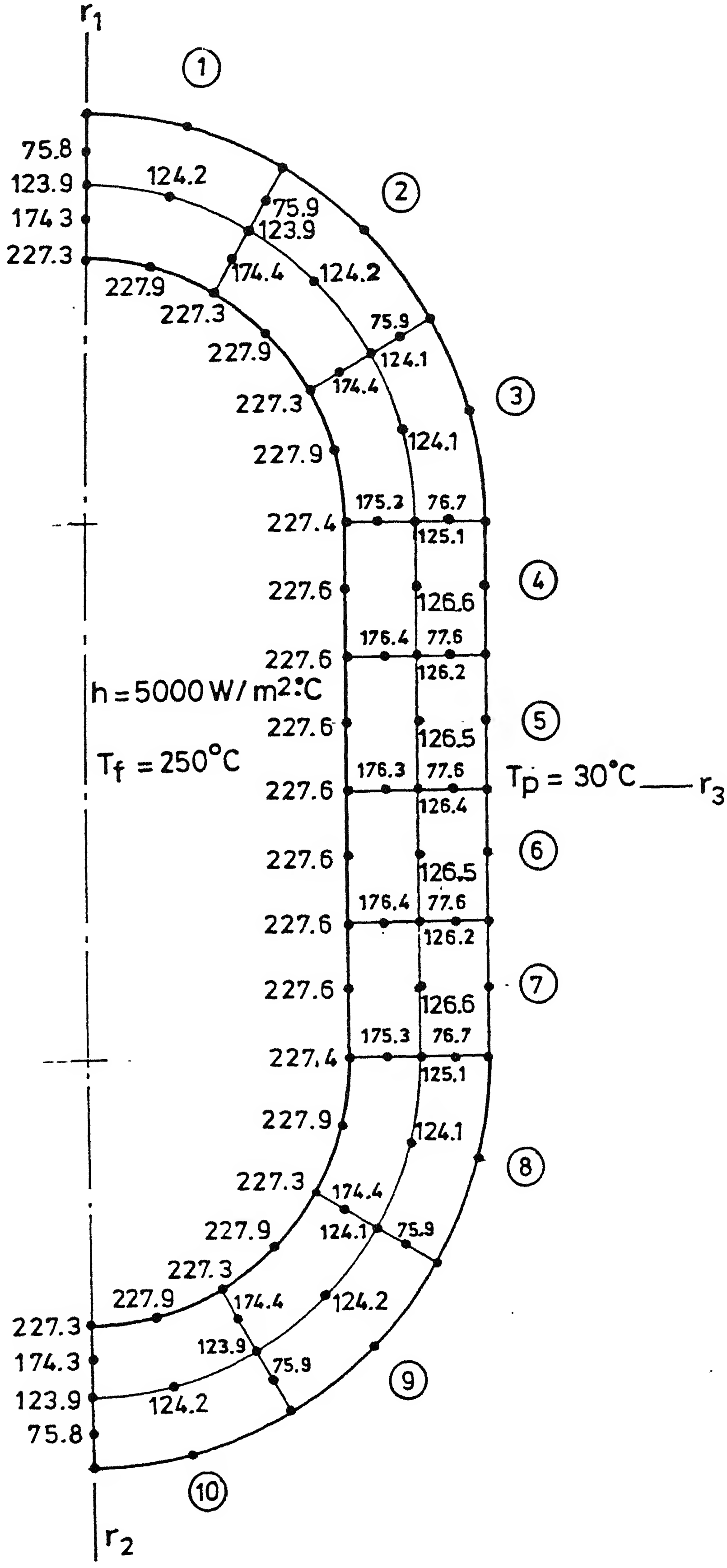


Fig. 4.2.2 Pressure vessel with inside surface exposed to fluid and outside surface temp. prescribed

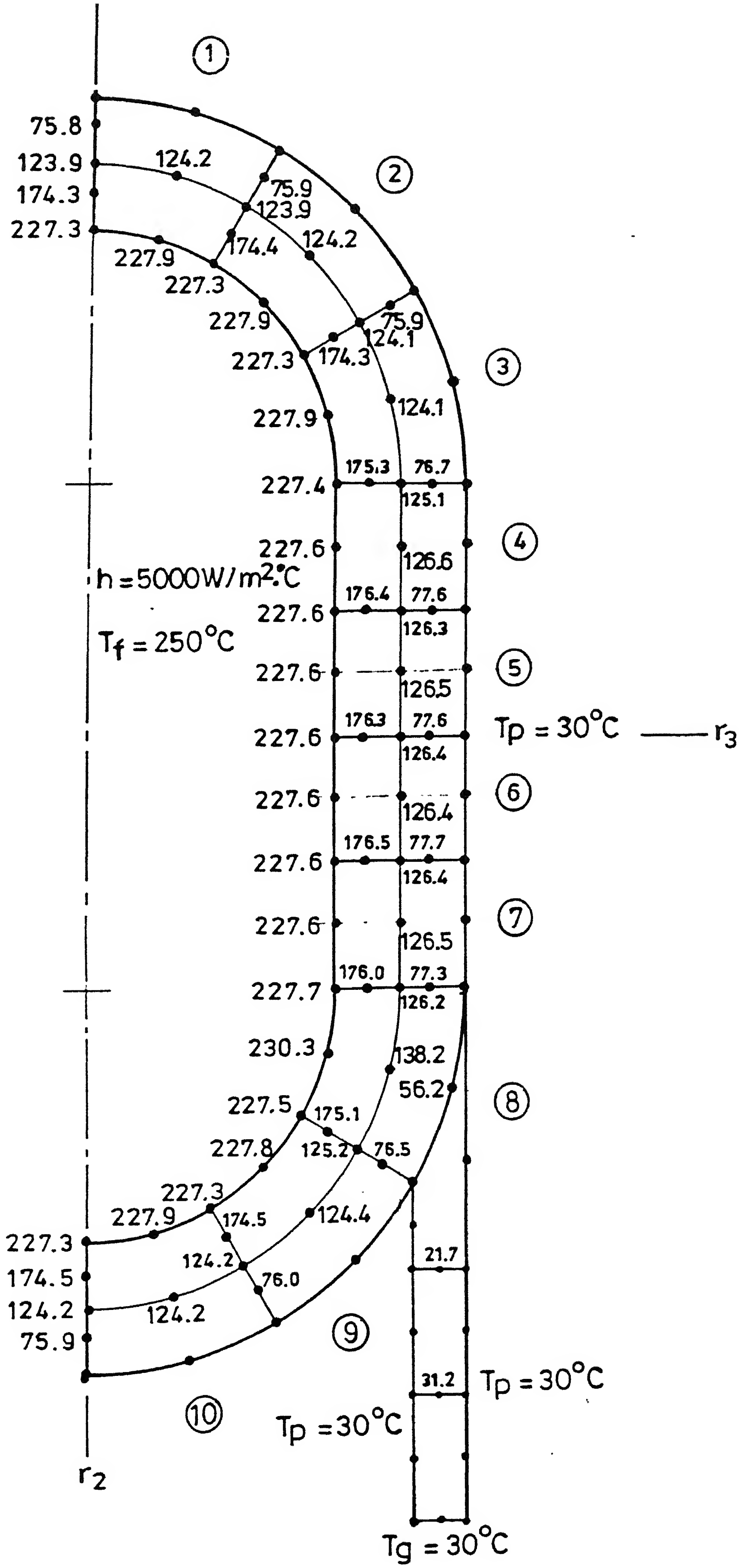
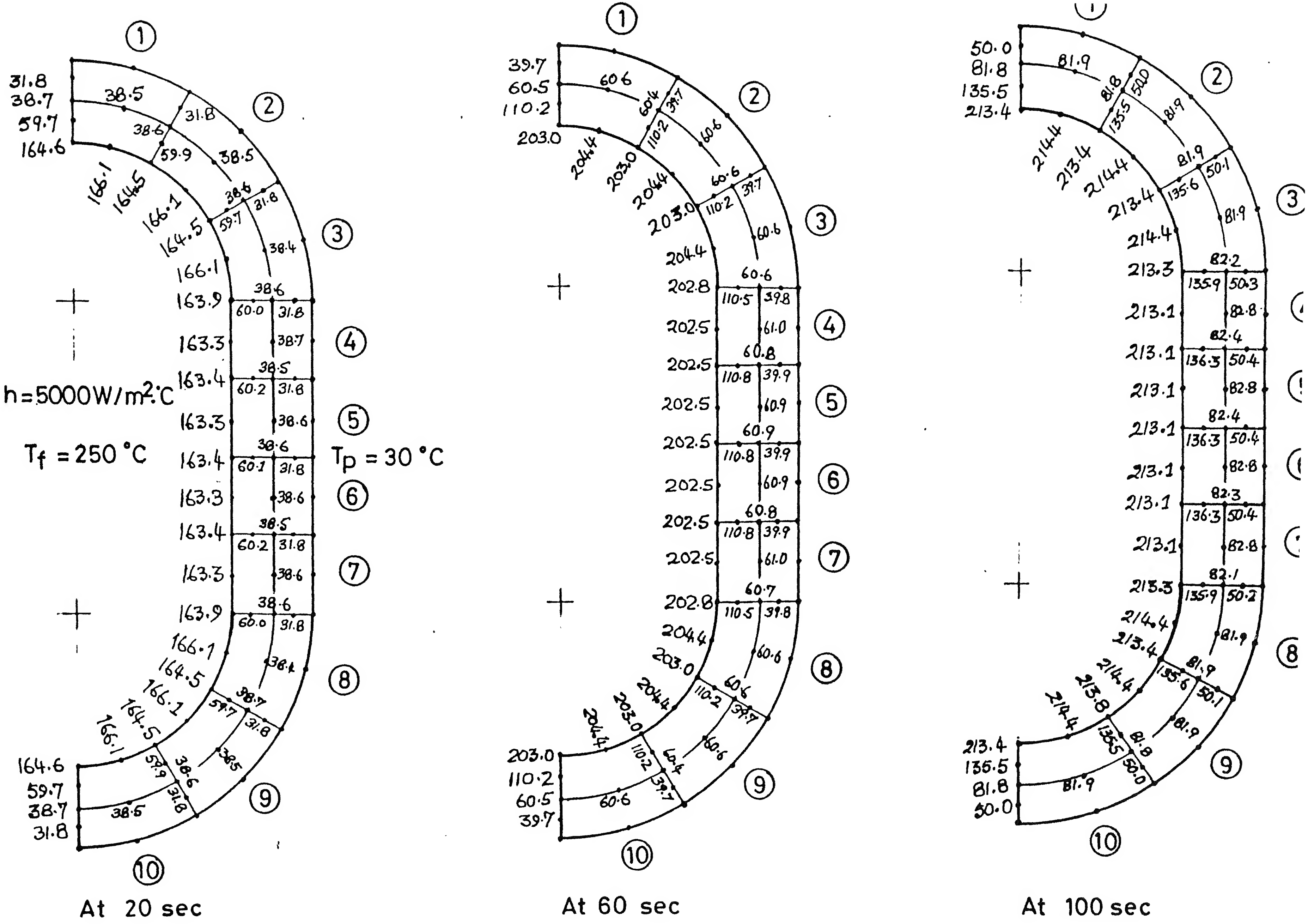


Fig. 4.2.3 Pressure vessel with skirt.



Temperatures during thermal transient analysis

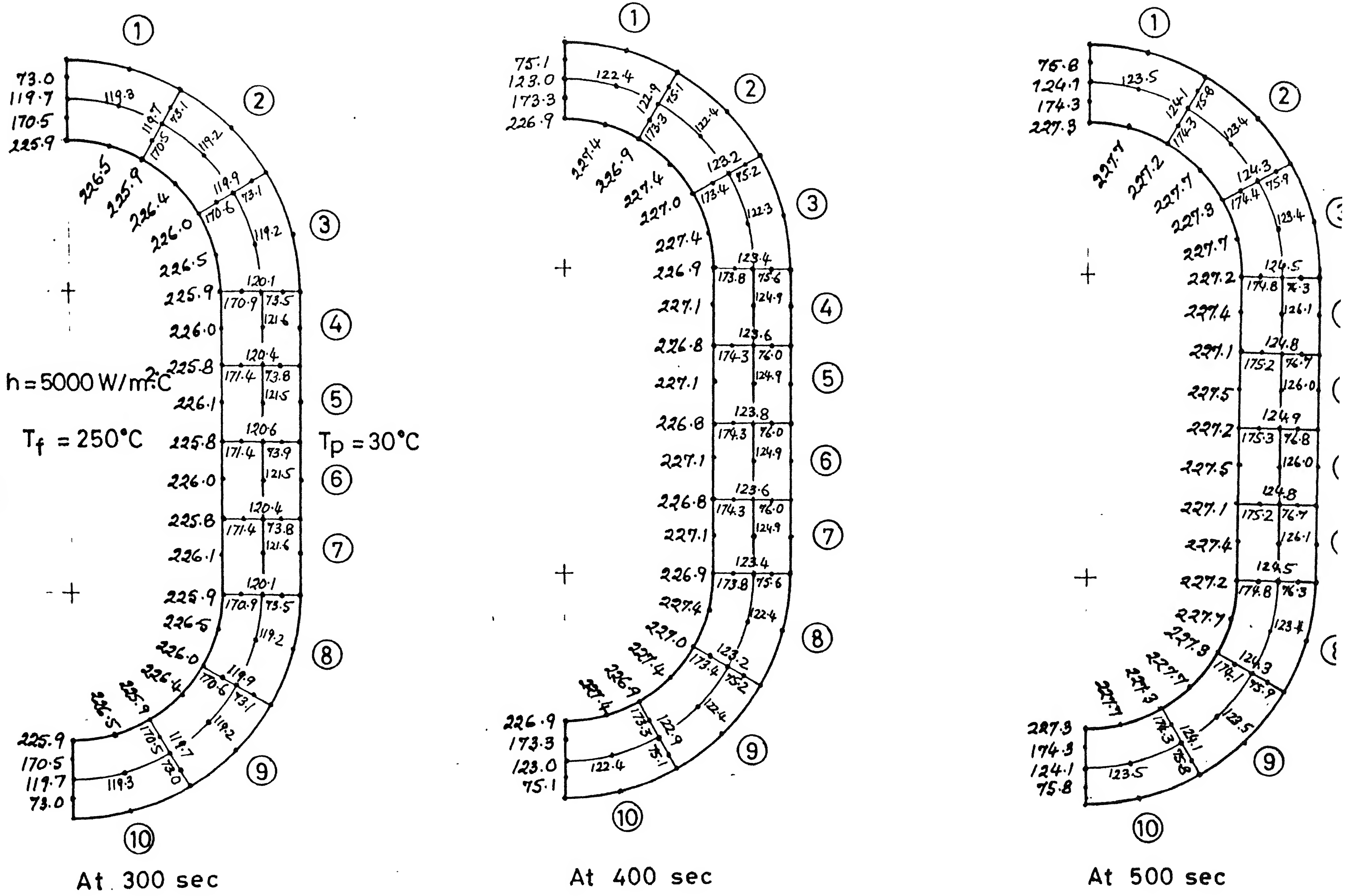


Fig 4.2.4(b) Temperatures during thermal transint analysis.

4.3 Stress analysis of pressure vessel with eight noded elements

Vertical pressure vessel of Fig. 4.2.1 is studied again for stress analysis. Here only half of the vessel is considered because of the double symmetry. The same should have been done for the thermal analysis, but it got overlooked there. Inside pressure is 10 MPa. vessel was divided into 5 strips with two eight noded elements in each strip.

Because of symmetry there will be no r-displacements along line r_1 and no z-displacements along the line r_3. This is shown in Fig. 4.2.5. Natural boundary conditions along line r_1 are that forces in z-direction are zero, and along line r_3 forces in the r-direction are zero. These geometrical and natural boundary conditions were used at the end of the multiplications of transfer matrices of all the strips.

Displacements and stresses are obtained by TFEM. Radial displacements are shown in Fig. 4.2.6. They show the expected trend. Radial displacements by TFEM along the line r_3 are .4754, .4685, .4623, .4562, .4504 mm and by analytical method for a straight shell [24] are .5411, .5362, .5316, .5275, .5238 mm. These results do not match very well as shell length is quite short and head geometry affects the displacements. Vertical displacements are shown in Fig. 4.2.7. These displacements also follow the expected trend.

Stresses are obtained at the Gauss points. Radial stresses are shown along the longitudinal gauss lines in Fig. 4.2.8. These values along fifteenth transverse gauss line are -9.814,

-7.195, -4.460, -4.597, -2.329, -.011 MPa and by analytical method for a straight shell [24] are -9.354, -7.223, -5.210, -4.085, -2.240, -.492. These results are comparable. Radial stresses along the first transerse gauss line are of the order of 50 MPa and compare well with the spherical shell theory value of 50 MPa. Hoop stresses along the six longitudinal gauss lines are shown in Fig. 4.2.9. They show the expected trend. These values along the fifteenth transverse gauss line are 105.838, 103.790, 101.987, 100.274, 98.505, 96.892 MPa and agree well with analytical method for a straight shell [24] values of 104.592, 102.461, 100.448, 99.323, 97.478, 95.731 MPa. Hoop stresses along the first transverse gauss line are of the order of 50 MPa and by analytical method for these spherical shell are also of the order of 50 MPa. σ_z-stresses along six longitudinal gauss lines are shown in Fig. 4.8.10. These also show the expected trend. These stresses along fifteenth transverse gauss line are of the order of 50 MPa and by approximate thin shell theory are also of the order of 50 MPa. Along first transverse gauss line, σ_z changes from -9.512 to -.069 MPa. This is also consistent as pressure inside is -10 MPa and 0 at outside.

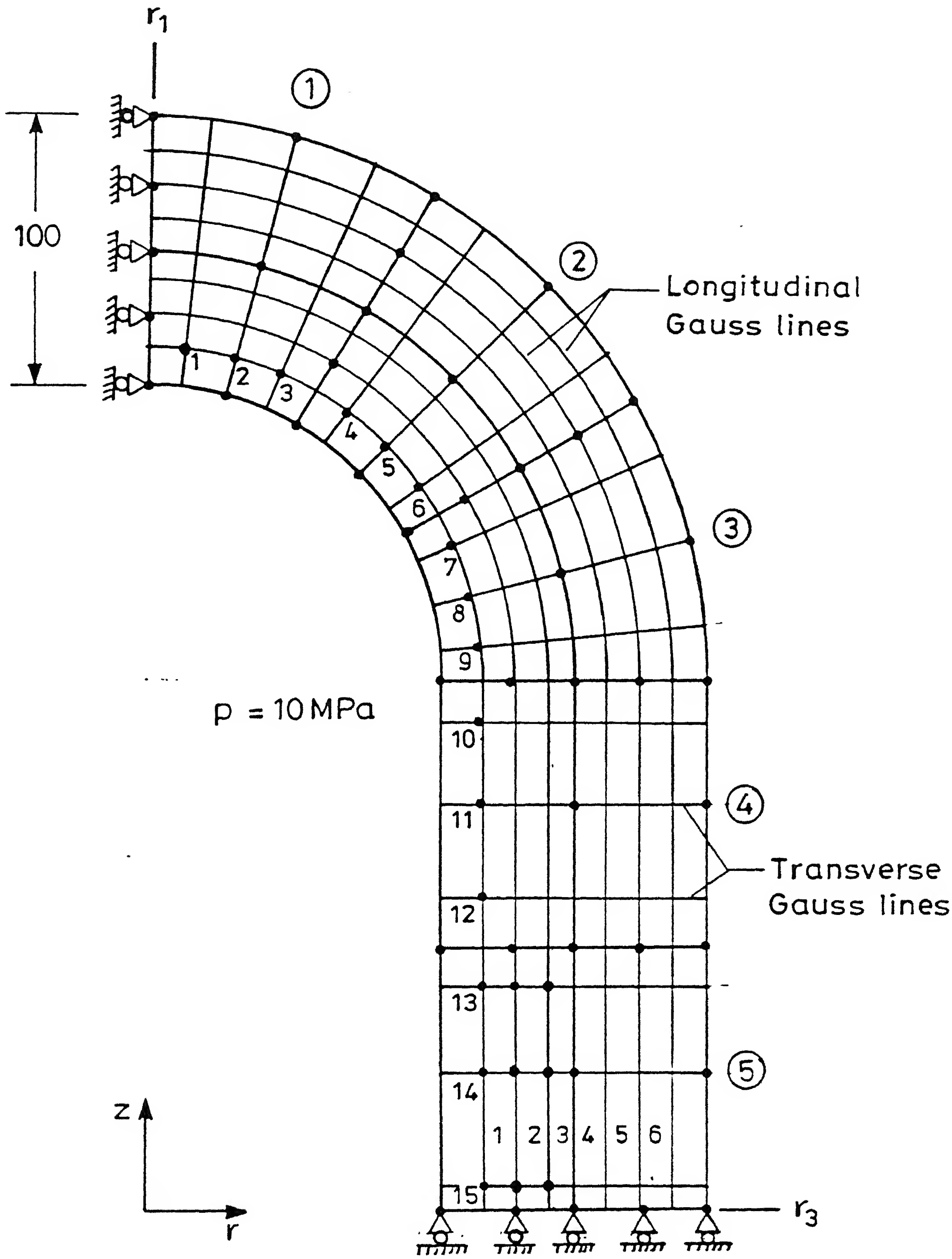


Fig. 4.2.5 Pressure vessel subjected to an internal pressure.

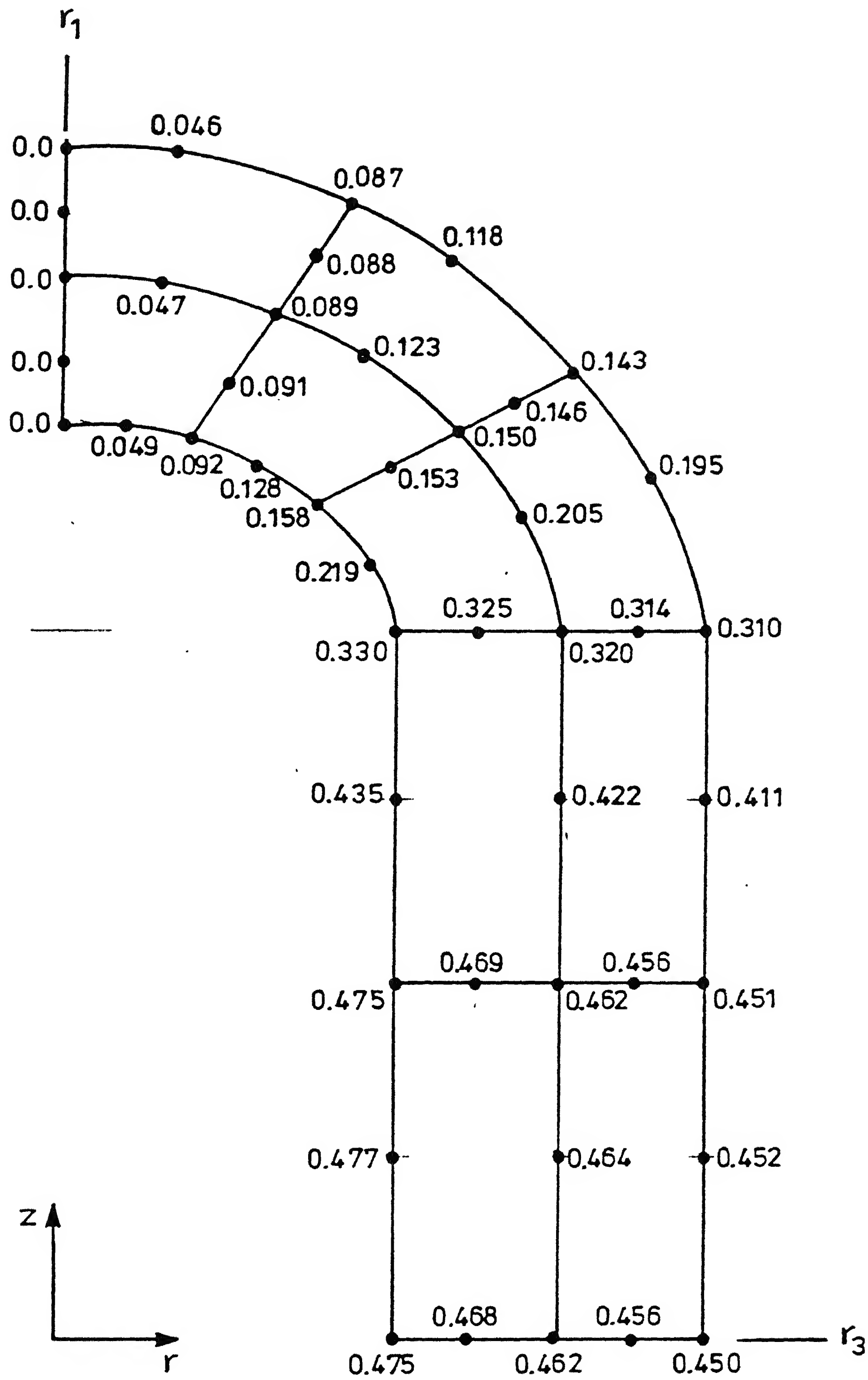


Fig. 4.2.6 r-displacements at the nodes.

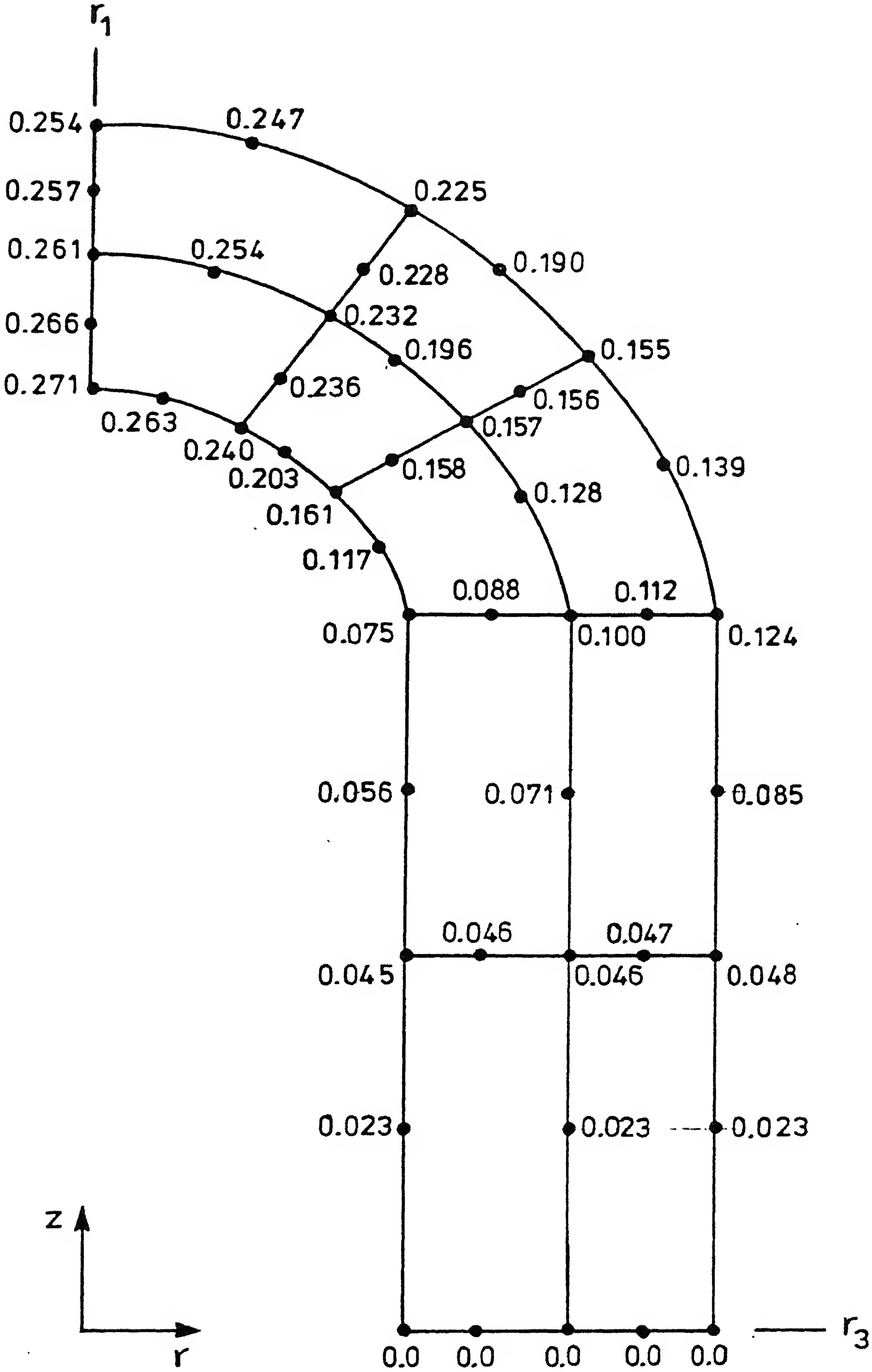


Fig. 4.2.7 z-displacements at the nodes.

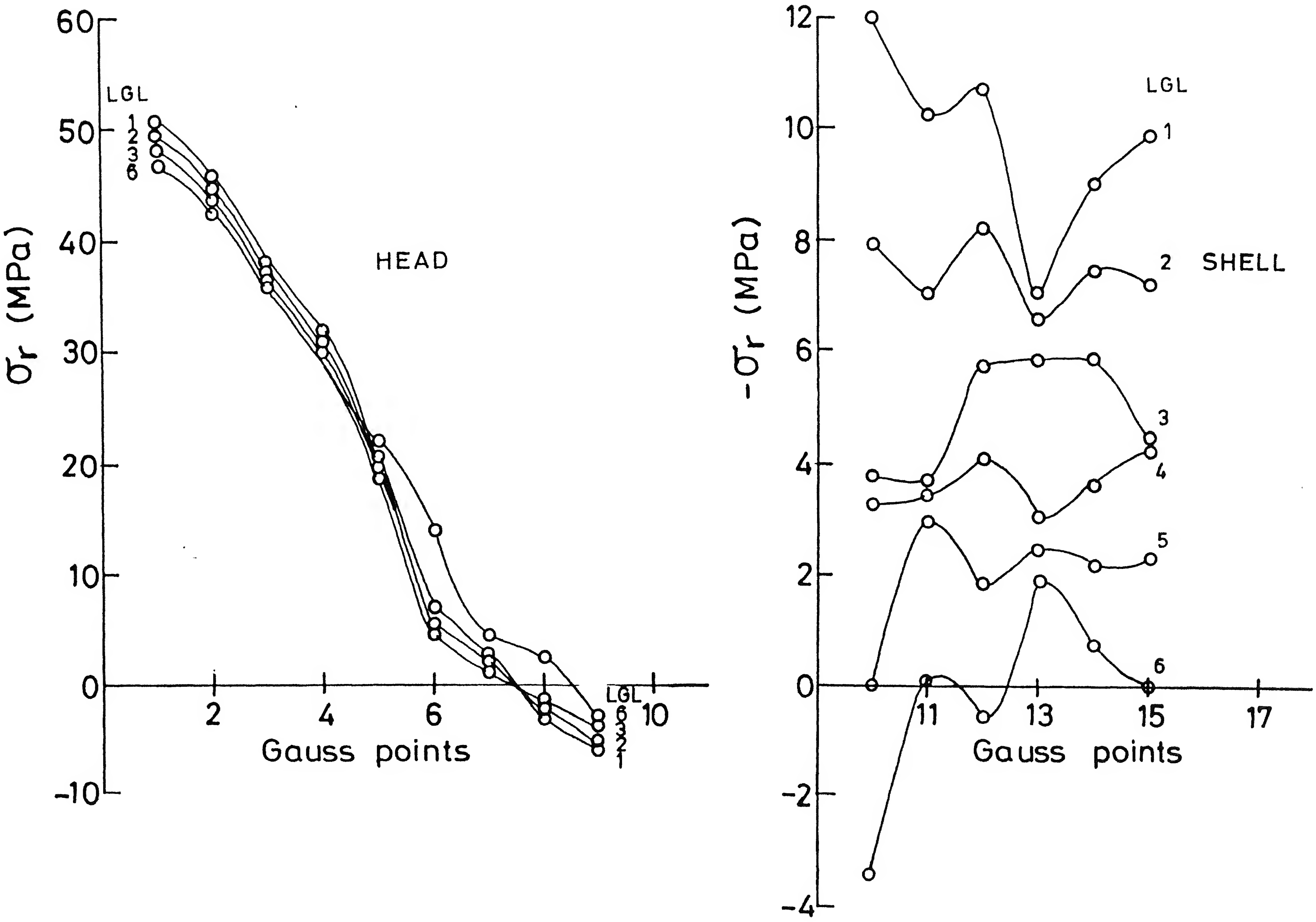


Fig 4.2.8 Radial stress vs Gauss points

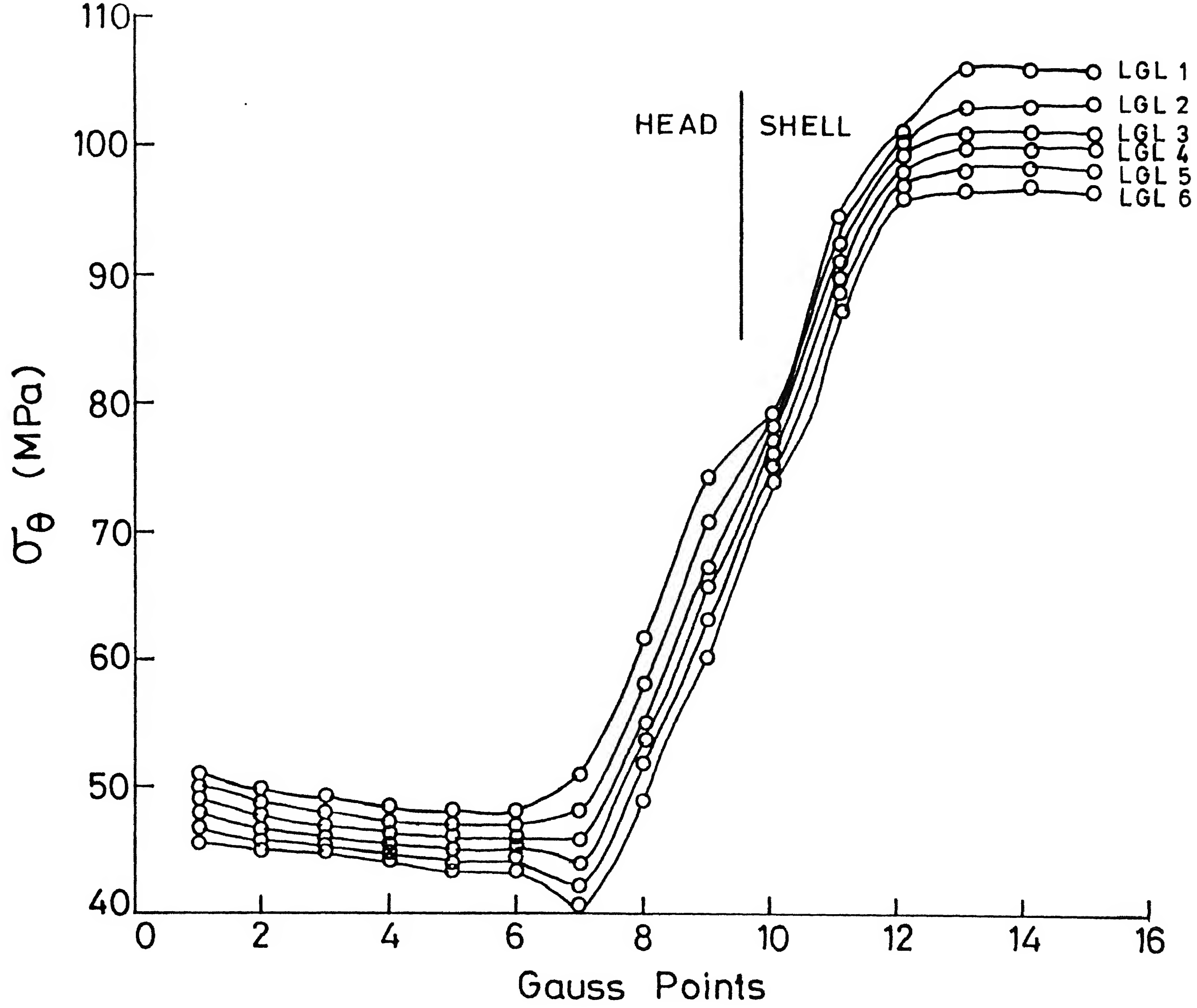


Fig 4.2.9 Circumferential stress vs Gauss points

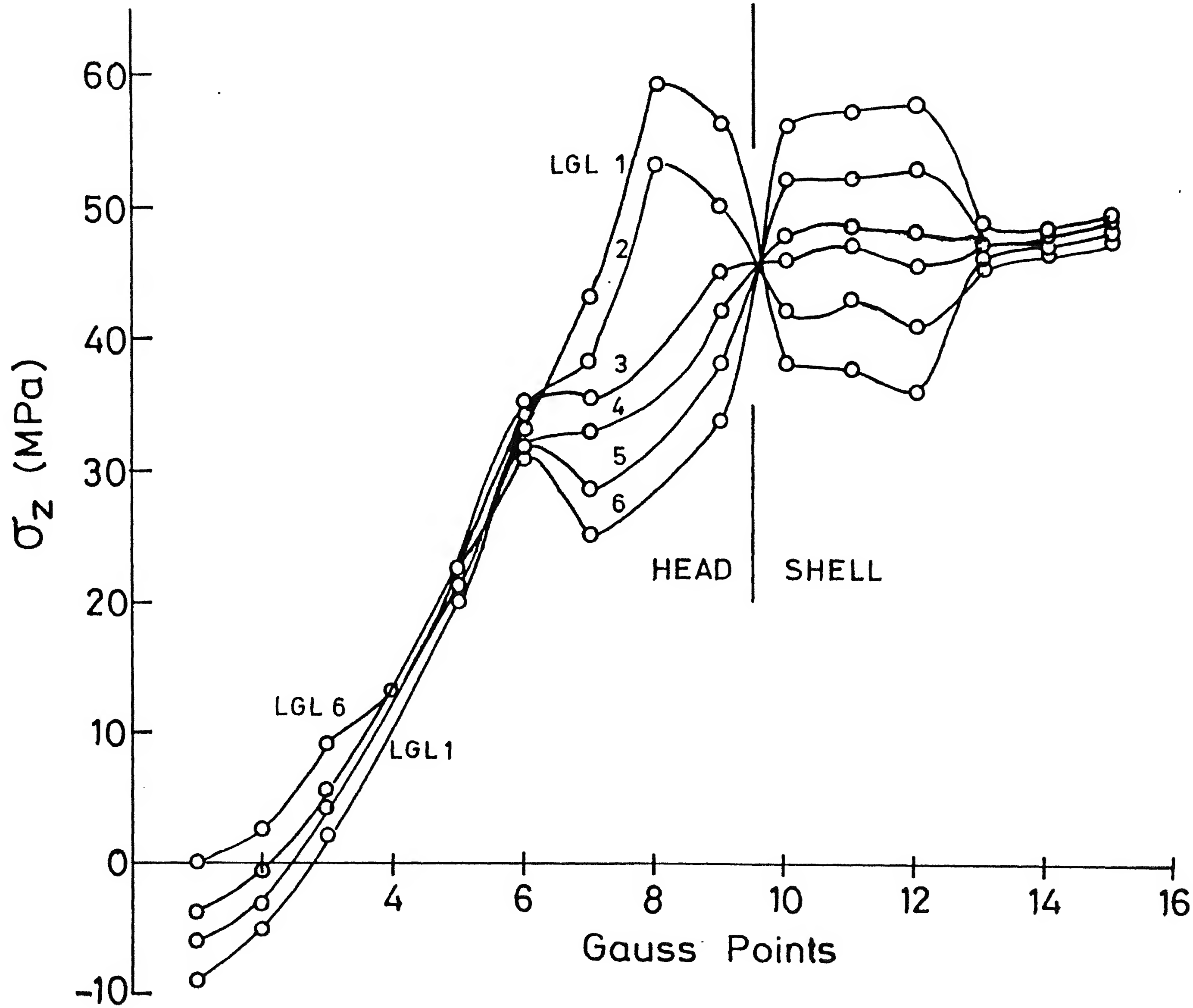


Fig 4.2.10 Longitudinal stress vs Gauss points

CHAPTER- 5

CONCLUSIONS

Following conclusions have been arrived at in the present work:

i) The TFEM equations can be obtained for elements having any number of nodes and any degrees of freedom.

ii) The transfer finite element method can be applied successfully to transient problems also.

iii) The transfer finite element method can be applied to pressure vessels having any geometry as long as the vessel is axi-symmetric.

iv) Results obtained by TFEM are accurate enough to match with the exact results.

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